VIBRATION PROBLEMS IN ENGINEERING

By

S. TIMOSHENKO

Professor of Theoretical and Engineering Mechanics
Stanford University

SECOND EDITION—FIFTH PRINTING

NEW YORK
D. VAN NOSTRAND COMPANY, Inc.
250 Fourth Avenue

First Published . . . October, 1928
Reprinted, September, 1929, August, 1935

Second Edition . . July, 1937
Reprinted, August, 1941, July, 1943, August, 1944, May, 1946

PRINTED IN THE U S A

PREFACE TO THE SECOND EDITION

In the preparation of the manuscript for the second edition of the book, the author's desire was not only to bring the book up to date by including some new material but also to make it more suitable for teaching purposes. With this in view, the first part of the book was entirely rewritten and considerably enlarged. A number of examples and problems with solutions or with answers were included, and in many places new material was added.

The principal additions are as follows: In the first chapter a discussion of forced vibration with damping not proportional to velocity is included, and an article on self-excited vibration. In the chapter on non-linear systems an article on the method of successive approximations is added and it is shown how the method can be used in discussing free and forced vibrations of systems with non-linear characteristics. The third chapter is made more complete by including in it a general discussion of the equation of vibratory motion of systems with variable spring characteristics. The fourth chapter, dealing with systems having several degrees of freedom, is also considerably enlarged by adding a general discussion of systems with viscous damping; an article on stability of motion with an application in studying vibration of a governor of a steam engine; an article on whirling of a rotating shaft due to hysteresis; and an article on the theory of damping vibration absorbers. There are also several additions in the chapter on torsional and lateral vibrations of shafts.

The author takes this opportunity to thank his friends who assisted in various ways in the preparation of the manuscript, and particularly Professor L. S. Jacobsen, who read over the complete manuscript and made many valuable suggestions, and Dr. J. A. Wojtaszak, who checked problems of the first chapter.

STEPHEN TIMOSHENKO

STANFORD UNIVERSITY,
May 29, 1937

PREFACE TO THE FIRST EDITION

With the increase of size and velocity in modern machines, the analysis of vibration problems becomes more and more important in mechanical engineering design. It is well known that problems of great practical significance, such as the balancing of machines, the torsional vibration of shafts and of geared systems, the vibrations of turbine blades and turbine discs, the whirling of rotating shafts, the vibrations of railway track and bridges under the action of rolling loads, the vibration of foundations, can be thoroughly understood only on the basis of the theory of vibration. Only by using this theory can the most favorable design proportions be found which will remove the working conditions of the machine as far as possible from the critical conditions at which heavy vibrations may occur.

In the present book, the fundamentals of the theory of vibration are developed, and their application to the solution of technical problems is illustrated by various examples, taken, in many cases, from actual experience with vibration of machines and structures in service. In developing this book, the author has followed the lectures on vibration given by him to the mechanical engineers of the Westinghouse Electric and Manufacturing Company during the year 1925, and also certain chapters of his previously published book on the theory of elasticity.*

The contents of the book in general are as follows:

The first chapter is devoted to the discussion of harmonic vibrations of systems with one degree of freedom. The general theory of free and forced vibration is discussed, and the application of this theory to balancing machines and vibration-recording instruments is shown. The Rayleigh approximate method of investigating vibrations of more complicated systems is also discussed, and is applied to the calculation of the whirling speeds of rotating shafts of variable cross-section.

Chapter two contains the theory of the non-harmonic vibration of systems with one degree of freedom. The approximate methods for investigating the free and forced vibrations of such systems are discussed. A particular case in which the flexibility of the system varies with the time is considered in detail, and the results of this theory are applied to the investigation of vibrations in electric locomotives with side-rod drive.

* *Theory of Elasticity*, Vol. II (1916)—St. Petersburg, Russia.

In chapter three, systems with several degrees of freedom are considered. The general theory of vibration of such systems is developed, and also its application in the solution of such engineering problems as: the vibration of vehicles, the torsional vibration of shafts, whirling speeds of shafts on several supports, and vibration absorbers.

Chapter four contains the theory of vibration of elastic bodies. The problems considered are: the longitudinal, torsional, and lateral vibrations of prismatical bars; the vibration of bars of variable cross-section; the vibrations of bridges, turbine blades, and ship hulls; the theory of vibration of circular rings, membranes, plates, and turbine discs.

Brief descriptions of the most important vibration-recording instruments which are of use in the experimental investigation of vibration are given in the appendix.

The author owes a very large debt of gratitude to the management of the Westinghouse Electric and Manufacturing Company, which company made it possible for him to spend a considerable amount of time in the preparation of the manuscript and to use as examples various actual cases of vibration in machines which were investigated by the company's engineers. He takes this opportunity to thank, also, the numerous friends who have assisted him in various ways in the preparation of the manuscript, particularly Messr. J. M. Lessells, J. Ormondroyd, and J. P. Den Hartog, who have read over the complete manuscript and have made many valuable suggestions.

He is indebted, also, to Mr. F. C. Wilharm for the preparation of drawings, and to the Van Nostrand Company for their care in the publication of the book.

S. Timoshenko

Ann Arbor, Michigan,
May 22, 1928.

CONTENTS

CHAPTER I

PAGE

CHAPTER II

CHAPTER III

CHAPTER IV

PAGE

CHAPTER V

CHAPTER VI

APPENDIX

CHAPTER I

HARMONIC VIBRATIONS OF SYSTEMS HAVING ONE DEGREE OF FREEDOM

1. Free Harmonic Vibrations. If an elastic system, such as a loaded beam, a twisted shaft or a deformed spring, be disturbed from its position of equilibrium by an impact or by the sudden application and removal of an additional force, the elastic forces of the member in the disturbed position will no longer be in equilibrium with the loading, and vibrations will ensue. Generally an elastic system can perform vibrations of different modes. For instance, a string or a beam while vibrating may assume the different shapes depending on the number of nodes subdividing the length of the member. In the simplest cases the configuration of the vibrating system can be determined by one quantity only. Such systems are called *systems having one degree of freedom.*

Let us consider the case shown in Fig. 1. If the arrangement be such that only vertical displacements of the weight W are possible and the mass of the spring be small in comparison with that of the weight W, the system can be considered as having *one degree of freedom.* The configuration will be determined completely by the vertical displacement of the weight.

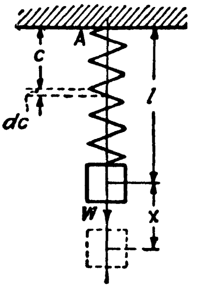

FIG. 1.

By an impulse or a sudden application and removal of an external force vibrations of the system can be produced. Such vibrations which are maintained by the elastic force in the spring alone are called *free* or *natural* vibrations. An analytical expression for these vibrations can be found from the differential equation of motion, which always can be written down if the forces acting on the moving body are known.

Let k denote the load necessary to produce a unit extension of the spring. This quantity is called *spring constant.* If the load is measured in pounds and extension in inches the spring constant will be obtained in lbs. per in. The static deflection of the spring under the action of the weight W will be

$$\delta_{st} = \frac{W}{k}.$$ k = L/E

Denoting a vertical displacement of the vibrating weight from its position of equilibrium by x and considering this displacement as positive if it is in a downward direction, the expression for the tensile force in the spring corresponding to any position of the weight becomes

$$F = W + kx. \tag{a}$$

In deriving the differential equation of motion we will use Newton's principle stating that the product of the mass of a particle and its acceleration is equal to the force acting in the direction of acceleration. In our case the mass of the vibrating body is W/g, where g is the acceleration due to gravity; the acceleration of the body W is given by the second derivative of the displacement x with respect to time and will be denoted by $\ddot{x}$; the forces acting on the vibrating body are the gravity force W, acting downwards, and the force F of the spring (Eq. a) which, for the position of the weight indicated in Fig. 1, is acting upwards. Thus the differential equation of motion in the case under consideration is

$$\frac{W}{g}\ddot{x} = W - (W + kx). \tag{1}$$

This equation holds for any position of the body W. If, for instance, the body in its vibrating motion takes a position above the position of equilibrium and such that a compressive force in the spring is produced the expression (a) becomes negative, and both terms on the right side of eq. (1) have the same sign. Thus in this case the force in the spring is added to the gravity force as it should be.

Introducing notation

$$p^2 = \frac{kg}{W} = \frac{g}{\delta_{st}}, \tag{2}$$

differential equation (1) can be represented in the following form

$$\ddot{x} + p^2 x = 0. \tag{3}$$

This equation will be satisfied if we put $x = C_1 \cos pt$ or $x = C_2 \sin pt$, where C_1 and C_2 are arbitrary constants. By adding these solutions the general solution of equation (3) will be obtained:

$$x = C_1 \cos pt + C_2 \sin pt. \tag{4}$$

It is seen that the vertical motion of the weight W has a vibratory character, since cos pt and sin pt are periodic functions which repeat themselves each time after an interval of time τ such that

$$p(\tau + t) - pt = 2\pi. \tag{b}$$

This interval of time is called the *period* of vibration. Its magnitude, from eq. (b), is

$$\tau = \frac{2\pi}{p} \tag{c}$$

or, by using notation (2),

$$\tau = 2\pi\sqrt{\frac{W}{kg}} = 2\pi\sqrt{\frac{\delta_{st}}{g}}. \tag{5}$$

It is seen that the period of vibration depends only on the magnitudes of the weight W and of the spring constant k and is independent of the magnitude of oscillations. We can say also that the period of oscillation of the suspended weight W is the same as that of a mathematical pendulum, the length of which is equal to the statical deflection δ_{st}. If the statical deflection δ_{st} is determined theoretically or experimentally the period τ can be calculated from eq. (5).

The number of cycles per unit time, say per second, is called the *frequency* of vibration. Denoting it by f we obtain

$$f = \frac{1}{\tau} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_{st}}}, \tag{6}$$

or, by substituting $g = 386$ in. per sec.^{2} and expressing δ_{st} in inches,

$$f = 3.127\sqrt{\frac{1}{\delta_{st}}} \text{ cycles per second.} \tag{6'}$$

A vibratory motion represented by equation (4) is called a *harmonic motion*. In order to determine the constants of integration C_1 and C_2, the initial conditions must be considered. Assume, for instance, that at the initial moment ($t = 0$) the weight W has a displacement x_0 from its position of equilibrium and that its initial velocity is $\dot{x}_0$. Substituting $t = 0$ in equation (4) we obtain

$$x_0 = C_1. \tag{d}$$

Taking now the derivative of eq. (4) with respect to time and substituting in this derivative $t = 0$, we have

$$\frac{\dot{x}_0}{p} = C_2 \tag{e}$$

Substituting in eq. (4) the values of the constants (*d*) and (*e*), the following expression for the vibratory motion of the weight W will be obtained:

$$x = x_0 \cos pt + \frac{\dot{x}_0}{p} \sin pt. \tag{7}$$

It is seen that in this case the vibration consists of two parts; a vibration which is proportional to $\cos pt$ and depends on the initial displacement of the system and another which is proportional to $\sin pt$ and depends on the

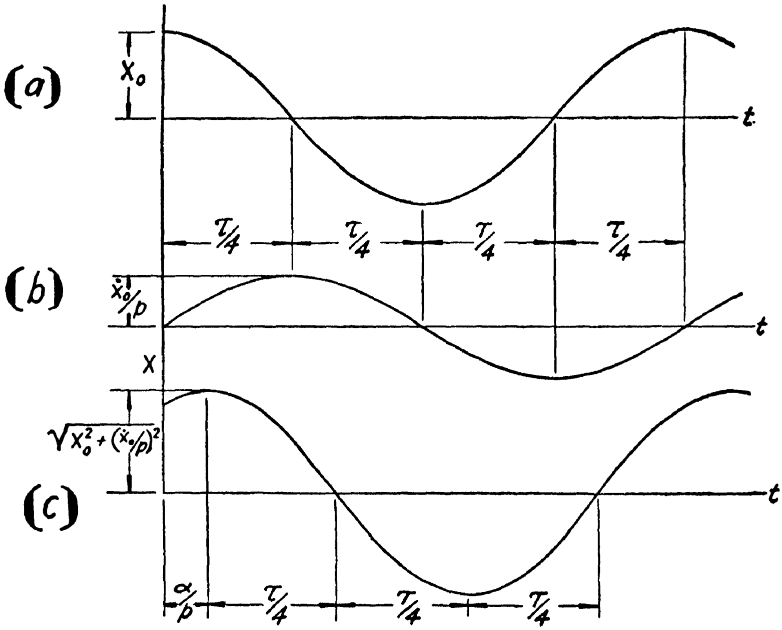

FIG. 2.

initial velocity $\dot{x}_0$. Each of these parts can be represented graphically, as shown in Figs. 2*a* and 2*b*, by plotting the displacements against the time. The total displacement x of the oscillating weight W at any instant t is obtained by adding together the ordinates of the two curves, (Fig. 2*a* and Fig. 2*b*) for that instant.

Another method of representing vibrations is by means of rotating vectors. Imagine a vector $\overline{OA}$, Fig. 3, of magnitude x_0 rotating with a constant angular velocity p around a fixed point, 0. This velocity is called *circular frequency* of vibration. If at the initial moment ($t = 0$) the vector coincides with x axis, the angle which it makes with the same axis at any instant t is equal to pt. The projection OA_1 of the vector on the x axis is equal to $x_0 \cos pt$ and represents the first term of expression (7). Taking now another vector $\overline{OB}$ equal to $\dot{x}_0/p$ and perpendicular to the vector $\overline{OA}$, its projection on the x axis gives the second term of expression (7). The total displacement x of the oscillating load W is obtained now by adding the projections on the x axis of the two perpendicular vectors $\overline{OA}$ and $\overline{OB}$, rotating with the angular velocity p.

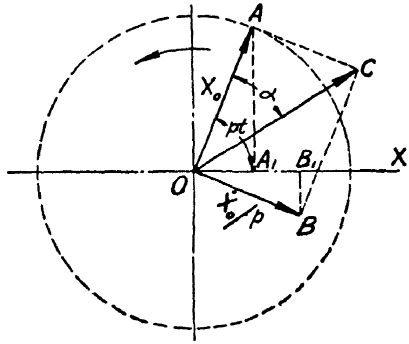

FIG. 3.

The same result will be obtained if, instead of vectors $\overline{OA}$ and $\overline{OB}$, we consider the vector $\overline{OC}$, equal to the geometrical sum of the previous two vectors, and take the projection of this vector on the x axis. The magnitude of this vector, from Fig. 3, is

$$\overline{OC} = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{p}\right)^2}$$

and the angle which it makes with the x axis is

$$pt - \alpha,$$

where

$$\alpha = \arctan \frac{\dot{x}_0}{px_0}.$$

Equating the projection of this vector on the x axis to expression (7) we obtain

$$\sqrt{x_0^2 + \left(\frac{\dot{x}_0}{p}\right)^2} \cos (pt - \alpha) = x_0 \cos pt + \frac{\dot{x}_0}{p} \sin pt. \tag{8}$$

It is seen that in this manner we added together the two simple harmonic motions, one proportional to $\cos pt$ and the other proportional to $\sin pt$. The result of this addition is a simple harmonic motion, proportional to $\cos (pt - \alpha)$, which is represented by Fig. 2c. The maximum ordinate of this curve, equal to $\sqrt{x_0^2 + (\dot{x}_0/p)^2}$, represents the maximum displacement of the vibrating body from the position of equilibrium and is called the *amplitude of vibration.*

Due to the angle α between the two rotating vectors $\overline{OA}$ and $\overline{OC}$ the maximum ordinate of the curve, Fig. 2c, is displaced with respect to the maximum ordinate of the curve, Fig. 2a, by the amount α/p. In such a case it may be said that the vibration, represented by the curve, Fig. 2c, is behind the vibration represented by the curve, Fig. 2a, and the angle α is called the *phase difference* of these two vibrations.

PROBLEMS

1. The weight $W = 30$ lbs. is vertically suspended on a steel wire of length $l = 50$ in. and of cross-sectional area $A = 0.001$ in.^{2}. Determine the frequency of free vibrations of the weight if the modulus for steel is $E = 30 \cdot 10^6$ lbs. per sq. in. Determine the amplitude of this vibration if the initial displacement $x_0 = 0.01$ in. and initial velocity $\dot{x}_0 = 1$ in. per sec.

Solution. Static elongation of the wire is $\delta_{st} = 30 \cdot 50/(30 \cdot 10^6 \cdot 0.001) = 0.05$ in. Then, from eq. (6′), $f = 3.13\sqrt{20} = 14.0$ sec.$^{-1}$. The amplitude of vibration, from eq. (8), is $\sqrt{x_0^2 + (\dot{x}_0/p)^2} = \sqrt{(0.01)^2 + [1/(2\pi \cdot 14)]^2} = .01513$ in.

2. Solve the previous problem assuming that instead of a vertical wire a helical spring is used for suspension of the load W. The diameter of the cylindrical surface containing the center line of the wire forming the spring is $D = 1$ in., the diameter of the wire $d = 0.1$ in., the number of coils $n = 20$. Modulus of material of the wire in shear $G = 12 \cdot 10^6$ lbs. per sq. in. In what proportion will the frequency of vibration be changed if the spring has 10 coils, the other characteristics of the spring remaining the same?

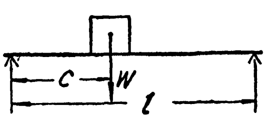

FIG. 4.

3. A load W is supported by a beam of length l, Fig. 4. Determine the spring constant and the frequency of free vibration of the load in the vertical direction neglecting the mass of the beam.

Solution. The statical deflection of the beam under load is

$$\delta_{st} = \frac{Wc^2(l-c)^2}{3lEI}.$$

Here c is the distance of the load from the left end of the beam and EI the flexural rigidity of the beam in the vertical plane. It is assumed that this plane contains one of the two principal axes of the cross section of the beam, so that vertical loads produce only vertical deflections. From the definition the spring constant in this case is

$$k = \frac{3lEI}{c^2(l-c)^2}.$$

Substituting δ_{st} in eq. (6) the required frequency can be calculated. The effect of the mass of the beam on the frequency of vibration will be discussed later, see Art. 16.

4. A load W is vertically suspended on two springs as shown in Fig. 5a. Determine the resultant spring constant and the frequency of vertical vibration of the load if the spring constants of the two springs are k_1 and k_2. Determine the frequency of vibration of the load W if it is suspended on two equal springs as shown in Fig. 5*b*.

Solution: In the case shown in Fig. 5*a* the statical deflection of the load W is

$$\delta_{st} = \frac{W}{k_1} + \frac{W}{k_2} = \frac{W(k_1 + k_2)}{k_1 k_2}.$$

The resultant spring constant is $k_1k_2/(k_1 + k_2)$. Substituting δ_{st} in eq. (6), the frequency of vibration becomes

$$f = \frac{1}{2\pi}\sqrt{\frac{gk_1k_2}{W(k_1 + k_2)}}.$$

In the case shown in Fig. 5*b*

$$\delta_{st} = \frac{W}{2k} \quad \text{and} \quad f = \frac{1}{2\pi}\sqrt{\frac{2gk}{W}}.$$

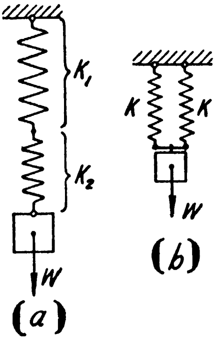

Fig. 5.

5. Determine the period of horizontal vibrations of the frame, shown in Fig. 6, supporting a load W applied at the center. The mass of the frame should be neglected in this calculation.

Solution. We begin with a statical problem and determine the horizontal deflection δ of the frame which a horizontal force H acting at the point of application of the load W will produce. Neglecting deformations due to tension and compression in the members

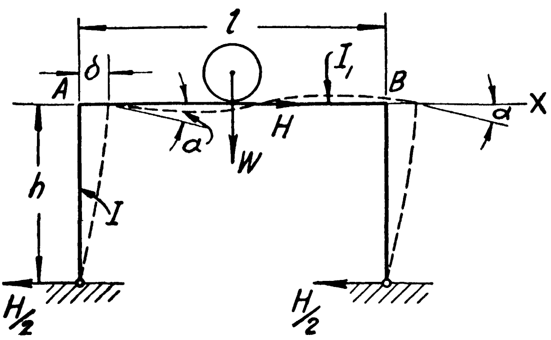

Fig. 6.

p 158
S. M.

and considering only bending, the horizontal bar AB is bent by two equal couples of magnitude $Hh/2$. Then the angle α of rotation of the joints A and B is

$$\alpha = \frac{Hhl}{12EI_1}.$$

Considering now the vertical members of the frame as cantilevers bent by the horizontal forces $H/2$, the horizontal deflection δ will consist of two parts, one due to bending of the

cantilevers and the second due to the rotation α of the joints A and B calculated above. Hence

$$\delta = \frac{Hh^3}{6EI} + \frac{Hh^2l}{12EI_1} = \frac{Hh^3}{6EI}\left(1 + \frac{1}{2}\frac{l}{h}\frac{I}{I_1}\right).$$

The spring constant in such case is

$$k = \frac{H}{\delta} = \frac{6EI}{h^3\left(1 + \frac{1}{2}\frac{l}{h}\frac{I}{I_1}\right)}.$$

Substituting in eq. (5), we obtain

$$\tau = 2\pi\sqrt{\frac{Wh^3\left(1 + \frac{1}{2}\frac{l}{h}\frac{I}{I_1}\right)}{6gEI}}.$$

If the rigidity of the horizontal member is large in comparison with the rigidity of the verticals, the term containing the ratio I/I_1 is small and can be neglected. Then

$$\tau = 2\pi\sqrt{\frac{Wh^3}{6gEI}}$$

and the frequency is

$$f = \frac{1}{2\pi}\sqrt{\frac{6gEI}{Wh^3}}.$$

6. Assuming that the load W in Fig. 1 represents the cage of an elevator moving down with a constant velocity v and the spring consists of a steel cable, determine the maximum stress in the cable if during motion the upper end A of the cable is suddenly stopped. Assume that the weight $W = 10{,}000$ lbs., $l = 60$ ft., the cross-sectional area of the cable $A = 2.5$ sq. in., modulus of elasticity of the cable $E = 15 \cdot 10^6$ lbs. per sq. in., $v = 3$ ft. per sec. The weight of the cable is to be neglected.

Solution. During the uniform motion of the cage the tensile force in the cable is equal to $W = 10{,}000$ lbs. and the elongation of the cable at the instant of the accident is $\delta_{st} = Wl/AE = .192$ in. Due to the velocity v the cage will not stop suddenly and will vibrate on the cable. Counting time from the instant of the accident, the displacement of the cage from the position of equilibrium at that instant is zero and its velocity is v. From eq. (7) we conclude that the amplitude of vibration will be equal to v/p, where $p = \sqrt{g/\delta_{st}} = 44.8$ sec.$^{-1}$ and $v = 36$ in. per sec. Hence the maximum elongation of the cable is $\delta_d = \delta_{st} + v/p = .192 + 36/44.8 = .192 + .803 = .995$ in. and the maximum stress is $(10{,}000/2.5)(.995/.192) = 20{,}750$ lbs. per sq. in. It is seen that due to the sudden stoppage of the upper end of the cable the stress in the cable increased in this case about five times.

7. Solve the previous problem assuming that a spring having a spring constant $k = 2000$ lbs. per in. is inserted between the lower end of the cable and the cage.

Solution. The statical deflection in this case is $\delta_{st} = .192 + 5 = 5.192$ in. and the amplitude of vibration, varying as square root of the statical deflection, becomes $.803\sqrt{5.192/.192}$. The maximum dynamical deflection is $5.192 + .803\sqrt{5.192/.192}$ and its ratio to the statical deflection is $1 + .803\sqrt{1/.192 \cdot 5.192} = 1.80$. Thus the maximum dynamical stress is (10,000/2.5)1.80 = 7,200 lbs. per sq. in. It is seen that by introducing the spring a considerable reduction in the maximum stress is obtained.

2. Torsional Vibration.—Let us consider a vertical shaft to the lower end of which a circular horizontal disc is attached, Fig. 7. If a torque is applied in the plane of the disc and then suddenly removed, free torsional vibration of the shaft with the disc will be produced. The angular position of the disc at any instant can be defined by the angle φ which a radius of the vibrating disc makes with the direction of the same radius when the disc is at rest. As the spring constant in this case we take the torque k which is necessary to produce an angle of twist of the shaft equal to one radian. In the case of a circular shaft of length l and diameter d we obtain from the known formula for the angle of twist

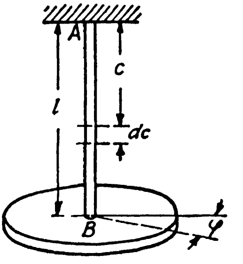

Fig. 7.

$$k = \frac{\pi d^4 G}{32l}. \tag{9}$$

For any angle of twist φ during vibration the torque in the shaft is $k\varphi$. The equation of motion in the case of a body rotating with respect to an immovable axis states that the moment of inertia of the body with respect to this axis multiplied with the angular acceleration is equal to the moment of the external forces acting on the body with respect to the axis of rotation. In our case this moment is equal and opposite to the torque $k\varphi$ acting on the shaft and the equation of motion becomes

$$I\ddot{\varphi} = -\,k\varphi \tag{a}$$

where I denotes the moment of inertia of the disc with respect to the axis of rotation, which in this case coincides with the axis of the shaft, and $\ddot{\varphi}$ is the angular acceleration of the disc. Introducing the notation

$$p^2 = \frac{k}{I}, \tag{10}$$

the equation of motion (a) becomes

$$\ddot{\varphi} + p^2\varphi = 0. \tag{11}$$

This equation has the same form as eq. (3) of the previous article, hence its solution has the same form as solution (7) and we obtain

$$\varphi = \varphi_0 \cos pt + \frac{\dot{\varphi}_0}{p} \sin pt, \tag{12}$$

where φ_0 and $\dot{\varphi}_0$ are the angular displacement and angular velocity respectively of the disc at the initial instant $t = 0$. Proceeding as in the previous article we conclude from eq. (12) that the period of torsional vibration is

$$\tau = \frac{2\pi}{p} = 2\pi \sqrt{\frac{I}{k}} \tag{13}$$

and its frequency is

$$f = \frac{1}{\tau} = \frac{1}{2\pi}\sqrt{\frac{k}{I}}. \tag{14}$$

In the case of a circular disc of uniform thickness and of diameter D,

$$I = \frac{WD^2}{8g},$$

where W is the weight of the disc. Substituting this in eqs. (13) and (14), and using expression (9), we obtain

$$\tau = 2\pi \sqrt{\frac{4WD^2l}{\pi g d^4 G}}, \quad f = \frac{1}{2\pi}\sqrt{\frac{\pi g d^4 G}{4WD^2 l}}. \tag{b}$$

It was assumed in our discussion that the shaft has a constant diameter d. When the shaft consists of parts of different diameters it can be readily reduced to an *equivalent shaft* having a constant diameter. Assume, for instance, that a shaft consists of two parts of lengths l_1 and l_2 and of diameters d_1 and d_2 respectively. If a torque M_t is applied to this shaft the angle of twist produced is

$$\varphi = \frac{32M_tl_1}{\pi d_1^4 G} + \frac{32M_tl_2}{\pi d_2^4 G} = \frac{32M_t}{\pi d_1^4 G}\left(l_1 + l_2\frac{d_1^4}{d_2^4}\right).$$

It is seen that the angle of twist of a shaft with two diameters d_1 and d_2 is the same as that of a shaft of constant diameter d_1 and of a reduced length L given by the equation

$$L = l_1 + l_2\frac{d_1^4}{d_2^4}.$$

The shaft of length L and diameter d_1 has the same spring constant as the given shaft of two different diameters and is an *equivalent* shaft in this case.

In general if we have a shaft consisting of portions with different diameters we can, without changing the spring constant of the shaft, replace any portion of the shaft of length l_n and of diameter d_n by a portion of a shaft of diameter d and of length l determined from the equation

$$l = l_n \frac{d^4}{d_n^{\,4}}. \tag{15}$$

The results obtained for the case shown in Fig. 7 can be used also in the case of a shaft with two rotating masses at the ends as shown in Fig. 8. Such a case is of practical importance since an arrangement of this kind may be encountered very often in machine design. A propeller shaft with the propeller on one end and the engine on the other is an example of this kind.* If two equal and opposite twisting couples are applied at the ends of the shaft in Fig. 8 and then suddenly removed, torsional vibrations will be produced during which the masses at the ends are always rotating in opposite directions.† From this fact it can be concluded at once that there is a certain intermediate cross section mn of the shaft which remains immovable during vibrations. This cross section is called the *nodal cross section*, and its position will be found from the condition that both portions of the shaft, to the right and to the left of the nodal cross section, must have the same period of vibration, since otherwise the condition that the masses at the ends always are rotating in opposite directions will not be fulfilled.

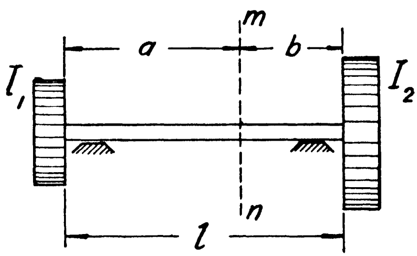

FIG. 8.

Applying eq. (13) to each of the two portions of the shaft we obtain

$$\sqrt{\frac{I_1}{k_1}} = \sqrt{\frac{I_2}{k_2}}, \quad \text{or} \quad \frac{k_1}{k_2} = \frac{I_1}{I_2}, \tag{c}$$

where k_1 and k_2 are the spring constants for the left and for the right portions of the shaft respectively. These quantities, as seen from eq. (9), are

* This is the case in which engineers for the first time found it of practical importance to go into investigation of vibrations, see H. Frahm, V.D.I., 1902, p. 797.

† This follows from the principle of moment of momentum. At the initial instant the moment of momentum of the two discs with respect to the axis of the shaft is zero and must remain zero since the moment of external forces with respect to the same axis is zero (friction forces are neglected). The equality to zero of moment of momentum requires that both masses rotate in opposite directions.

inversely proportional to the lengths of the corresponding portions of the shaft and from eq. (c) follows

$$\frac{a}{b} = \frac{I_2}{I_1}$$

and, since $a + b = l$, we obtain

$$a = \frac{lI_2}{I_1 + I_2}, \quad b = \frac{lI_1}{I_1 + I_2} \tag{d}$$

Applying now to the left portion of the shaft eqs. (13) and (14) we obtain

$$\tau = 2\pi\sqrt{\frac{I_1}{k_1}} = 2\pi\sqrt{\frac{32lI_1I_2}{\pi d^4G(I_1 + I_2)}} \tag{16}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{\pi d^4G(I_1 + I_2)}{32lI_1I_2}}. \tag{17}$$

From these formulae the period and the frequency of torsional vibration can be calculated provided the dimensions of the shaft, the modulus G and the moments of inertia of the masses at the ends are known. The mass of the shaft is neglected in our present discussion and its effect on the period of vibration will be considered later, see Art. 16.

It can be seen (eq. d) that if one of the rotating masses has a very large moment of inertia in comparison with the other the nodal cross section can be taken at the larger mass and the system with two masses (Fig. 8) reduces to that with one mass (Fig. 7).

PROBLEMS

1. Determine the frequency of torsional vibration of a shaft with two circular discs of uniform thickness at the ends, Fig. 8, if the weights of the discs are $W_1 = 1000$ lbs. and $W_2 = 2000$ lbs. and their outer diameters are $D_1 = 50$ in. and $D_2 = 75$ in. respectively. The length of the shaft is $l = 120$ in. and its diameter $d = 4$ in. Modulus in shear $G = 12 \cdot 10^6$ lbs. per sq. in.

Solution. From eqs. (d) the distance of the nodal cross section from the larger disc is

$$a = \frac{120 \cdot 1000 \cdot 50^2}{1000 \cdot 50^2 + 2000 \cdot 75^2} = \frac{120}{1 + 4.5} = 21.8 \text{ in.}$$

Substituting in eq. (b) we obtain

$$f = \frac{1}{2\pi}\sqrt{\frac{\pi \cdot 386 \cdot 4^4 \cdot 12 \cdot 10^6}{4 \cdot 2000 \cdot 75^2 \cdot 21.8}} = 9.80 \text{ oscillations per sec.}$$

2. In what proportion will the frequency of vibration of the shaft considered in the previous problem increase if along a length of 64 in. the diameter of the shaft will be increased from 4 in. to 8 in.

Solution. The length of 64 in. of 8 in. diameter shaft can be replaced by a 4 in. length of 4 in. diameter shaft. Thus the length of the equivalent shaft is $4 + 56 = 60$ in., which is only one-half of the length of the shaft considered in the previous problem. Since the frequency of vibration is inversely proportional to the square root of the length of the shaft (see eq. 17), we conclude that as the result of the reinforcement of the shaft its frequency increases in the ratio $\sqrt{2} : 1$.

3. A circular bar fixed at the upper end and supporting a circular disc at the lower end (Fig. 7) has a frequency of torsional vibration equal to $f = 10$ oscillations per second. Determine the modulus in shear G if the length of the bar $l = 40$ in., its diameter $d = 0.5$ in., the weight of the disc $W = 10$ lbs., and its outer diameter $D = 12$ in.

Solution. From eq. (*b*), $G = 12 \cdot 10^6$ lbs. per sq. in.

4. Determine the frequency of vibration of the ring, Fig. 9, about the axis 0, assuming that the center of the ring remains fixed and that rotation of the rim is accompanied

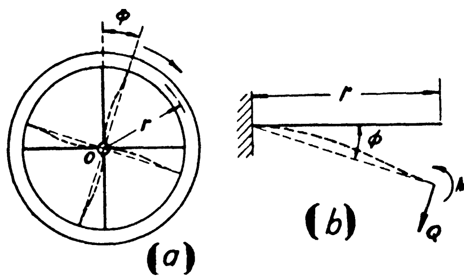

FIG. 9.

by some bending of the spokes indicated in the figure by dotted lines. Assume that the total mass of the ring is distributed along the center line of the rim and take the length of the spokes equal to the radius r of this center line. Assume also that the bending of the rim can be neglected so that the tangents to the deflection curves of the spokes have radial directions at the rim. The total weight of the ring W and the flexural rigidity B of spokes are given.

Solution. Considering each spoke as a cantilever of length r, Fig. 9*b*, at the end of which a shearing force Q and a bending moment M are acting and using the known formulas for bending of a cantilever, the following expressions for the slope φ and the deflection $r\varphi$ at the end are obtained

$$\varphi = \frac{Qr^2}{2B} - \frac{Mr}{B}, \quad r\phi = \frac{Qr^3}{3B} - \frac{Mr^2}{2B},$$

from which

$$M = \frac{Qr}{3} = \frac{2B\phi}{r}.$$

If M_t denotes the torque applied to the rim we have

$$M_t = 4Qr - 4M = \frac{16B\phi}{r}.$$

The torque required to produce an angle of rotation of the rim equal to one radian is the spring constant and is equal to $k = 16B/r$. Substituting in eq. (14), we obtain the required frequency

$$f = \frac{1}{2\pi}\sqrt{\frac{16B}{rI}} = \frac{1}{2\pi}\sqrt{\frac{16gB}{Wr^3}}.$$

3. Forced Vibrations.—In the two previous articles free vibrations of systems with one degree of freedom have been discussed. Let us consider now the case when in addition to the force of gravity and to the force in the spring (Fig. 1) there is acting on the load W a periodical *disturbing* force $P \sin \omega t$. The period of this force is $\tau_1 = 2\pi/\omega$ and its frequency is $f_1 = \omega/2\pi$. Proceeding as before (see p. 2) we obtain the following differential equation

$$\frac{W}{g}\ddot{x} = W - (W + kx) + P \sin \omega t, \tag{a}$$

or, by using eq. (2) and notation

$$q = \frac{Pg}{W}, \tag{b}$$

we obtain

$$\ddot{x} + p^2 x = q \sin \omega t. \tag{18}$$

A particular solution of this equation is obtained by assuming that x is proportional to $\sin \omega t$, i.e., by taking

$$x = A \sin \omega t, \tag{c}$$

where A is a constant, the magnitude of which must be chosen so as to satisfy eq. (18). Substituting (c) in that equation we find

$$A = \frac{q}{p^2 - \omega^2}.$$

Thus the required particular solution is

$$x = \frac{q \sin \omega t}{p^2 - \omega^2}.$$

Adding to this particular solution expression (4), representing the solution of the eq. (3) for free vibration, we obtain

$$x = C_1 \cos pt + C_2 \sin pt + \frac{q \sin \omega t}{p^2 - \omega^2}. \tag{19}$$

This expression contains two constants of integration and represents the general solution of the eq. (18). It is seen that this solution consists of two parts, the first two terms represent free vibrations which were discussed before and the third term, depending on the disturbing force, represents the *forced vibration* of the system. It is seen that this later vibration has the same period $\tau_1 = 2\pi/\omega$ as the disturbing force has. Its amplitude A, is equal to the numerical value of the expression

$$\frac{q}{p^2 - \omega^2} = \frac{P}{k} \frac{1}{1 - \omega^2/p^2}. \tag{20}$$

The factor P/k is the deflection which the maximum disturbing force P would produce if acting statically and the factor $1/(1 - \omega^2/p^2)$ takes care of the dynamical action of this force. The absolute value of this factor is usually called the *magnification factor*. We see that it depends only on the

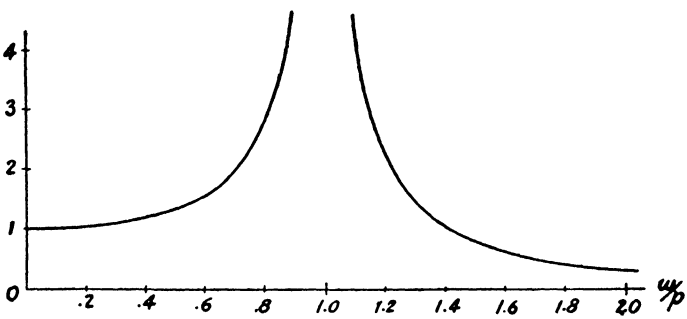

FIG. 10.

ratio ω/p which is obtained by dividing the frequency of the disturbing force by the frequency of free vibration of the system. In Fig. 10 the values of the magnification factor are plotted against the ratio ω/p.

It is seen that for the small values of the ratio ω/p, i.e., for the case when the frequency of the disturbing force is small in comparison with the frequency of free vibration, the magnification factor is approximately unity, and deflections are about the same as in the case of a statical action of the force P.

When the ratio ω/p approaches unity the magnification factor and the amplitude of forced vibration rapidly increase and become infinite for $\omega = p$, i.e., for the case when the frequency of the disturbing force exactly coincides with the frequency of free vibration of the system. This is the *condition of resonance*. The infinite value obtained for the amplitude of forced vibrations indicates that if the pulsating force acts on the vibrating system always at a proper time and in a proper direction the amplitude of vibration increases indefinitely provided there is no damping. In practical problems we always have damping the effect of which on the amplitude of forced vibration will be discussed later (see Art. 9).

When the frequency of the disturbing force increases beyond the frequency of free vibration the magnification factor again becomes finite. Its absolute value diminishes with the increase of the ratio ω/p and approaches zero when this ratio becomes very large. This means that when a pulsating force of high frequency (ω/p is large) acts on the vibrating body it produces vibrations of very small amplitude and in many cases the body may be considered as remaining immovable in space. The practical significance of this fact will be discussed in the next article.

Considering the sign of the expression $1/(1 - \omega^2/p^2)$ it is seen that for the case $\omega < p$ this expression is positive and for $\omega > p$ it becomes negative. This indicates that when the frequency of the disturbing force is less than that of the natural vibration of the system the forced vibrations and the disturbing force are always in the same *phase*, i.e., the vibrating load (Fig. 1) reaches its lowest position at the same moment that the disturbing force assumes its maximum value in a downward direction. When $\omega > p$ the difference in phase between the forced vibration and the disturbing force becomes equal to π. This means that at the moment when the force is a maximum in a downward direction the vibrating load reaches its upper position. This phenomenon can be illustrated by the following simple experiment. In the case of a simple pendulum AB (Fig. 11) forced vibrations can be produced by giving an oscillating motion in the horizontal direction to the point A. If this oscillating motion has a frequency lower than that of the pendulum the extreme positions of the pendulum during such vibrations will be as shown in Fig. 11–a, the motions of the points A and B will be in the same phase. If the oscillatory motion of the point A has a higher frequency than that of the pendulum the extreme positions of the pendulum during vibration will be as shown in Fig. 11–b. The phase difference of the motions of the points A and B in this case is equal to π.

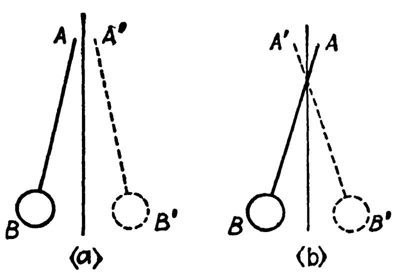

FIG. 11.

In the above discussion the disturbing force was taken proportional to $\sin \omega t$. The same conclusions will be obtained if $\cos \omega t$, instead of $\sin \omega t$, be taken in the expression for the disturbing force.

In the foregoing discussion the third term only of the general solution (19) has been considered. In applying a disturbing force, however, not only forced vibrations are produced but also free vibrations given by the first two terms in expression (19). After a time the latter vibrations will be damped out due to different kinds of resistance * but at the beginning of motion they may be of practical importance. The amplitude of the free vibration can be found from the general solution (19) by taking into consideration the initial conditions. Let us assume that at the initial instant ($t = 0$) the displacement and the velocity of the vibrating body are equal to zero. The arbitrary constants of the solution (19) must then be determined in such a manner that for $t = 0$

$$x = 0 \quad \text{and} \quad \dot{x} = 0.$$

These conditions will be satisfied by taking

$$C_1 = 0, \quad C_2 = -\frac{q\omega/p}{p^2 - \omega^2}.$$

Substituting in expression (19), we obtain

$$x = \frac{q}{p^2 - \omega^2}\left(\sin \omega t - \frac{\omega}{p}\sin pt\right). \tag{21}$$

Thus the motion consists of two parts, free vibration proportional to sin pt and forced vibration proportional to sin ωt.

Let us consider the case when the frequency of the disturbing force is very close to the frequency of free vibrations of the system, i.e., ω is close to p. Using notation

$$p - \omega = 2\Delta,$$

where Δ is a small quantity, and neglecting a small term with the factor $2\Delta/p$, we represent expression (21) in the following form:

$$x = \frac{q}{p^2 - \omega^2}(\sin \omega t - \sin pt) = \frac{2q}{p^2 - \omega^2}\cos\frac{(\omega + p)t}{2}\sin\frac{(\omega - p)t}{2}$$

$$= -\frac{2q \sin \Delta t}{p^2 - \omega^2}\cos\frac{(\omega + p)t}{2} \approx -\frac{q \sin \Delta t}{2\omega\Delta}\cos \omega t. \tag{22}$$

Since Δ is a small quantity the function sin Δt varies slowly and its period, equal to $2\pi/\Delta$, is large. In such a case expression (22) can be considered as

* Damping was entirely neglected in the derivation of eq. (18).

representing vibrations of a period $2\pi/\omega$ and of a variable amplitude equal to $q \sin \Delta t/2\omega\Delta$. This kind of vibration is called *beating* and is shown in Fig. 12. The period of beating, equal to $2\pi/\Delta$, increases as ω approaches p,

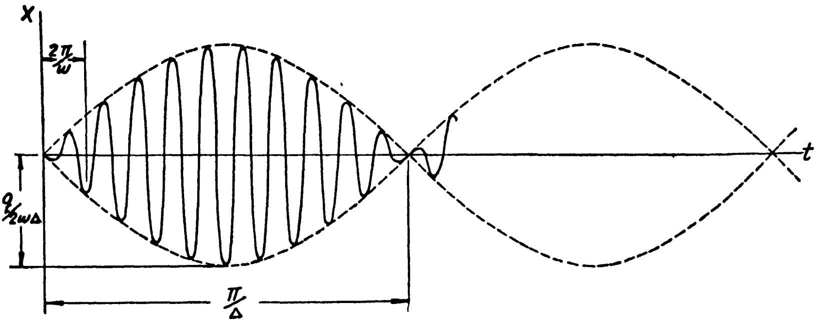

Fig. 12.

i.e., as we approach the condition of resonance. For limiting condition $\omega = p$ we can put in expression (22) Δt, instead of $\sin \Delta t$ and we obtain

$$x = -\frac{qt}{2\omega} \cos \omega t. \tag{23}$$

The amplitude of vibration in eq. (23) increases indefinitely with the time as shown in Fig. 13.

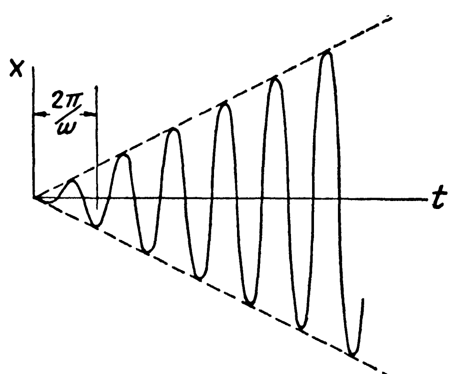

Fig. 13.

PROBLEMS

1. A load W suspended vertically on a spring, Fig. 1, produces a statical elongation of the spring equal to 1 inch. Determine the magnification factor if a vertical disturbing force $P \sin \omega t$, having the frequency 5 cycles per sec. is acting on the load. Determine the amplitude of forced vibration if $W = 10$ lbs., $P = 2$ lbs.

Solution. From eq. (2), $p = \sqrt{g/\delta_{st}} = \sqrt{386} = 19.6$ sec.$^{-1}$. We have also $\omega = 2\pi \cdot 5 = 31.4$ sec.$^{-1}$. Hence the magnification factor is $1/(\omega^2/p^2 - 1) = 1/1.56$. Deflection produced by P if acting statically is 0.2 in. and the amplitude of forced vibration is $0.2/1.56 = 0.128$ in.

2. Determine the total displacement of the load W of the previous problem at the instant $t = 1$ sec. if at the initial moment ($t = 0$) the load is at rest in equilibrium position.

Answer. $x = -\dfrac{.2}{1.56}\left(\sin 10\pi - \dfrac{31.4}{19.6}\sin 19.6\right) = +\ .14$ inch.

3. Determine the amplitude of forced torsional vibration of a shaft in Fig. 7 produced by a pulsating torque $M \sin \omega t$ if the free torsional vibration of the same shaft has the frequency $f = 10$ sec.$^{-1}$, $\omega = 10\pi$ sec.$^{-1}$ and the angle of twist produced by torque M, if acting on the shaft statically, is equal to .01 of a radian.

Solution. Equation of motion in this case is (see Art. 2)

$$\ddot{\varphi} + p^2\varphi = \frac{M}{I}\sin \omega t,$$

where φ is the angle of twist and $p^2 = k/I$. The forced vibration is

$$\varphi = \frac{M}{I(p^2 - \omega^2)}\sin \omega t = \frac{M}{k(1 - \omega^2/p^2)}\sin \omega t.$$

Noting that the statical deflection is $M/k = 0.01$ and $p = 2\pi \cdot 10$ we obtain the required amplitude equal to

$$\frac{0.01}{(1 - \frac{1}{4})} = 0.0133 \text{ radian.}$$

4. Instruments for Investigating Vibrations.—For measuring vertical vibrations a weight W suspended on a spring can be used (Fig. 14). If the point of suspension A is immovable and a vibration in the vertical direction of the weight is produced, the equation of motion (1) can be applied, in which x denotes displacement of W from the position of equilibrium. Assume now that the box, containing the suspended weight W, is attached to a body performing vertical vibration. In such a case the point of suspension A vibrates also and due to this fact forced vibration of the weight will be produced. Let us assume that vertical vibrations of the box are given by equation

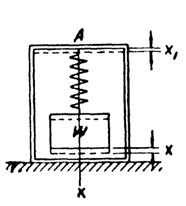

FIG. 14.

$$x_1 = a \sin \omega t, \tag{a}$$

so that the point of suspension A performs simple harmonic motion of amplitude a. In such case the elongation of the spring is $x - x_1$ and the corresponding force in the spring is $k(x - x_1)$. The equation of motion of the weight then becomes

$$\frac{W}{g}\ddot{x} = -k(x - x_1),$$

or, by substituting for x_1 its expression (a) and using notations

$$p^2 = \frac{kg}{W}, \quad \frac{akg}{W} = q, \tag{b}$$

we obtain

$$\ddot{x} + p^2 x = q \sin \omega t.$$

This equation coincides with equation (18) for forced vibrations and we can apply here the conclusions of the previous article. Assuming that the free vibrations of the load are damped out and considering only forced vibrations, we obtain

$$x = \frac{q \sin \omega t}{p^2 - \omega^2} = \frac{a \sin \omega t}{1 - \omega^2/p^2}. \tag{c}$$

It is seen that in the case when ω is small in comparison with p, i.e., the frequency of oscillation of the point of suspension A is small in comparison with the frequency of free vibration of the system, the displacement x is approximately equal to x_1 and the load W performs practically the same oscillatory motion as the point of suspension A does. When ω approaches p the denominator in expression (c) approaches zero and we approach resonance condition at which heavy forced vibrations are produced.

Considering now the case when ω is very large in comparison with p, i.e., the frequency of vibration of the body to which the instrument is attached is very high in comparison with frequency of free vibrations of the load W the amplitude of forced vibrations (c) becomes small and the weight W can be considered as immovable in space. Taking, for instance, $\omega = 10p$ we find that the amplitude of forced vibrations is only $a/99$, i.e., in this case vibrations of the point of suspension A will scarcely be transmitted to the load W.

This fact is utilized in various instruments used for measuring and recording vibrations. Assume that a dial is attached to the box with its plunger pressing against the load W as shown in Fig. 209. During vibration the hand of the dial, moving back and forth, gives the double amplitude of relative motion of the weight W with respect to the box. This amplitude is equal to the maximum value of the expression

$$x - x_1 = a \sin \omega t \left(\frac{1}{1 - \omega^2/p^2} - 1 \right)$$

$$= a \sin \omega t \cdot \frac{\omega^2/p^2}{1 - \omega^2/p^2}. \tag{24}$$

When p is small in comparison with ω this value is very close to the amplitude a of the vibrating body to which the instrument is attached. The numerical values of the last factor in expression (24) are plotted against the ratio ω/p in Fig. 18.

The instrument described has proved very useful in power plants for studying vibrations of turbo-generators. Introducing in addition to vertical also horizontal springs, as shown in Fig. 209, the horizontal vibrations also can be measured by the same instrument. The springs of the instrument are usually chosen in such a manner that the frequencies of free vibrations of the weight W both in vertical and horizontal directions are about 200 per minute. If a turbo-generator makes 1800 revolutions per minute it can be expected that, owing to some unbalance, vibrations of the foundation and of the bearings of the same frequency will be produced. Then the dials of the instrument attached to the foundation or to a bearing will give the amplitudes of vertical and horizontal vibrations with sufficient accuracy since in this case $\omega/p = 9$ and the difference between the motion in which we are interested and the relative motion (24) is a small one.

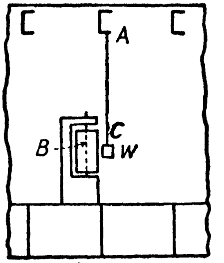

Fig. 15.

To get a record of vibrations a cylindrical drum rotating with a constant speed can be used. If such a drum with vertical axis is attached to the box, Fig. 14, and a pencil attached to the weight presses against the drum, a complete record of the relative motion (24) during vibration will be recorded. On this principle various vibrographs are built, for instance, the vibrograph constructed by the Cambridge Instrument Company, shown in Fig. 213 and Geiger's vibrograph, shown in Fig. 214. A simple arrangement for recording vibrations in ship hulls is shown in Fig. 15. A weight W is attached at point A to a beam by a rubber band AC. During vertical vibrations of the hull this weight remains practically immovable provided the period of free vibrations of the weight is sufficiently large.

Then the pencil attached to it will record the vibrations of the hull on a rotating drum B. To get a satisfactory result the frequency of free vibrations of the weight must be small in comparison with that of the hull of the ship. This requires that the statical elongation of the string AC must be large. For instance, to get a frequency of ½ of an oscillation per second the elongation of the string under the statical action of the weight W must be nearly 3 ft. The requirement of large extensions is a defect in this type of instrument.

A device analogous to that shown in Fig. 14 can be applied also for measuring accelerations. In such a case a rigid spring must be used and the frequency of natural vibrations of the weight W must be made very large in comparison with the frequency of the vibrating body to which the instrument is attached. Then p is large in comparison with ω in expression (24) and the relative motion of the load W is approximately equal to $a\omega^2 \sin \omega t/p^2$ and proportional to the acceleration $\ddot{x}_1$ of the body to which the instrument is attached. Due to the rigidity of the spring the relative displacements of the load W are usually small and require special devices for recording them. An electrical method for such recording, used in investigating accelerations of vibrating parts in electric locomotives, is discussed later (see page 459).

PROBLEMS

1. A wheel is rolling along a wavy surface with a constant horizontal speed v, Fig. 16. Determine the amplitude of the forced vertical vibrations of the load W attached to

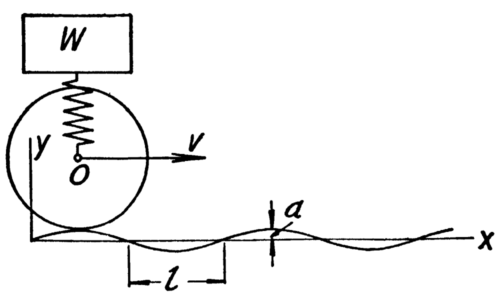

FIG. 16

the axle of the wheel by a spring if the statical deflection of the spring under the action of the load W is $\delta_{st} = 3.86$ ins., $v = 60$ ft. per sec. and the wavy surface is given by the equation $y = a \sin \frac{\pi x}{l}$ in which $a = 1$ in. and $l = 36$ in.

Solution. Considering vertical vibrations of the load W on the spring we find, from eq. 2, that the square of the circular frequency of these vibrations is $p^2 = g/\delta_{st} = 100$.

Due to the wavy surface the center o of the rolling wheel makes vertical oscillations. Assuming that at the initial moment $t = 0$ the point of contact of the wheel is at $x = 0$ and putting $x = vt$, these vertical oscillations are given by the equation $y = a \sin \frac{\pi v t}{l}$. The forced vibration of the load W is now obtained from equation (c) substituting in it $a = 1$ in., $\omega = \frac{\pi v}{l} = 20\pi$, $p^2 = 100$. Then the amplitude of forced vibration is $1/(4\pi^2 - 1) = .026$ in. At the given speed v the vertical oscillations of the wheel are transmitted to the load W only in a very small proportion. If we take the speed v of the wheel $\frac{1}{4}$ as great we get $\omega = 5\pi$ and the amplitude of forced vibration becomes $1/(\pi^2/4 - 1) = 0.68$ in. By further decrease in speed v we finally come to the condition of resonance when $\pi v/l = p$ at which condition heavy vibration of the load W will be produced.

2. For measuring vertical vibrations of a foundation the instrument shown in Fig. 14 is used. What is the amplitude of these vibrations if their frequency is 1800 per minute, the hand of the dial fluctuates between readings giving deflections .100 in. and .120 in. and the springs are chosen so that the statical deflection of the weight W is equal to 1 in.?

Solution. From the dial reading we conclude that the amplitude of relative motion, see eq. 24, is .01 in. The frequency of free vibrations of the weight W, from eq. (6), is $f = 3.14$ per sec. Hence $\omega/p = 30/3.14$. The amplitude of vibration of the foundation, from eq. 24, is

$$a = .01 \frac{(30/3.14)^2 - 1}{(30/3.14)^2} = .00989 \text{ in.}$$

3. A device such as shown in Fig. 14 is used for measuring vertical acceleration of a cab of a locomotive which makes, by moving up and down, 3 vertical oscillations per second. The spring of the instrument is so rigid that the frequency of free vibrations of the weight W is 60 per second. What is the maximum acceleration of the cab if the vibrations recorded by the instrument representing the relative motion of the weight W with respect to the box have an amplitude $a_1 = 0.001$ in.? What is the amplitude a of vibration of the cab?

Solution. From eq. 24 we have

$$a_1 = \frac{a\omega^2}{p^2 - \omega^2}.$$

Hence the maximum vertical acceleration of the cab is

$$a\omega^2 = a_1(p^2 - \omega^2).$$

Noting that $p = 2\pi \cdot 60$ and $\omega = 2\pi \cdot 3$, we obtain

$$a\omega^2 = .001 \cdot 4\pi^2(60^2 - 3^2) = 142 \text{ in. sec.}^{-2}$$

and

$$a = \frac{142}{36\pi^2} = .4 \text{ in.}$$

5. Spring Mounting of Machines.—Rotating machines with some unbalance produce on their foundations periodic disturbing forces as a result of which undesirable vibrations of foundations and noise may occur. To reduce these bad effects a spring mounting of machines is sometimes used. Let a block of weight W in Fig. 17 represent the machine and P denote the cent 'fugal force due to unbalance when the angular velocity of the machine one radian per second. Then at any angular velocity ω the centrifugal force is $P\omega^2$ and, measuring the angle of rotation as shown in the figure, we obtain the vertical and the horizontal components of the disturbing force equal to $P\omega^2 \sin \omega t$ and $P\omega^2 \cos \omega t$ respectively. If the machine is rigidly attached to a rigid foundation, as shown in Fig. 17a, there will be no motion of the block W and the total centrifugal force will

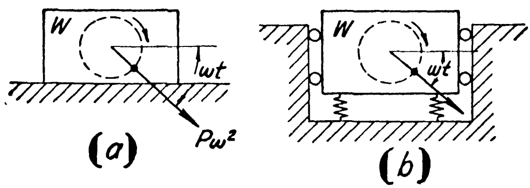

FIG. 17.

be transmitted to the foundation. To diminish the force acting on the foundation, let us introduce a spring mounting, as shown in Fig. 17b, and assume that there is a constraint preventing lateral movements of the machine. In this way a vibrating system consisting of the block W on vertical springs, analogous to the system shown in Fig. 1, is obtained. To determine the pulsating vertical force transmitted through the springs to the foundation the vertical vibration of the block under the action of the disturbing force $P\omega^2 \sin \omega t$ must be investigated.* Using the expression for forced vibrations given in article 3 and substituting $P\omega^2$ for P, we find that the amplitude of forced vibration is equal to the numerical value of the expression

$$\frac{Pp^2}{k} \cdot \frac{\omega^2/p^2}{1 - \omega^2/p^2}. \tag{a}$$

Where k is the spring constant, i.e., the force required to produce vertical deflection of the block equal to unity, and p^2 is defined by eq. 2. A similar

* It is assumed here that vibrations are small and do not effect appreciably the magnitude of the disturbing force calculated on the assumption that the unbalanced weight is rotating about fixed axis.

expression has been obtained before in discussing the theory of vibrographs, see eq. 24. It is seen that for a given value of the ratio Pp^2/k the amplitude of forced vibration depends only on the value of the ratio ω/p. The absolute values of the second factor in expression (a) are plotted against the values of ω/p in Fig. 18. It is seen that for large values of ω/p these quantities approach unity and the absolute value of expression (a) approaches Pp^2/k. Having the amplitude of forced vibration of the block W and multiplying it by the spring constant k, we obtain the maximum pulsating force in the spring which will be transmitted to the foundation. Keeping in mind that $P\omega^2$ is the maximum vertical disturbing force when the machine is rigidly attached to the foundation, Fig. 17a, it can be concluded from (a) that the spring mounting reduces the disturbing force

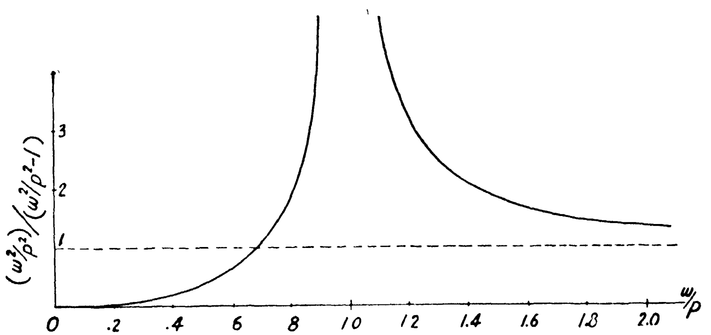

FIG. 18.

only if $1 - \omega^2/p^2$ is numerically larger than one, i.e., when $\omega > p\sqrt{2}$. When ω is very large in comparison with p, i.e., when the machine is mounted on soft springs, expression (a) approaches numerically the value Pp^2/k and we have, due to spring mounting, a reduction of the vertical disturbing force in the ratio p^2/ω^2. From this discussion we see that to reduce disturbing forces transmitted to foundation the machine must be mounted on soft springs such that the frequency of free vibration of the block W is small in comparison with the number of revolutions per second of the machine. The effect of damping in supporting springs will be discussed later (see Art. 10). To simplify the problem we have discussed here only vertical vibrations of the block. To reduce the horizontal disturbing force horizontal springs must be introduced and horizontal vibrations must be investigated. We will again come to the conclusion that the frequency of vibration must be small in comparison with the number of revolutions per second of the machine in order to reduce horizontal disturbing forces.

PROBLEMS

1. A machine of weight $W = 1000$ lbs. and making 1800 revolutions per minute is supported by four helical springs (Fig. 17*b*) made of steel wire of diameter $d = \frac{1}{2}$ in. The diameter corresponding to the center line of the helix is $D = 4$ in. and the number of coils $n = 10$. Determine the maximum vertical disturbing force transmitted to the foundation if the centrifugal force of unbalance for the angular speed equal to 1 radian per sec. is $P = 1$ pound.

Solution. The statical deflection of the springs under the action of the load W is

$$\delta_{st} = \frac{2nD^3W}{d^4G} = \frac{2 \cdot 10 \cdot 4^3 \cdot 1000}{(\frac{1}{2})^4 \cdot 12 \cdot 10^6} = 1.71 \text{ in.}$$

from which the spring constant $k = 1000/1.71 = 585$ lbs. per in. and the square of the circular frequency of free vibration $p^2 = g/\delta_{st} = 225$ are obtained. By using equation (*a*) we obtain the maximum force transmitted to foundation

$$1 \cdot \frac{(60\pi)^2}{(60\pi)^2/(225) - 1} = 227 \text{ lbs.}$$

2. In what proportion will the vertical disturbing force of the previous problem increase if instead of 4 there will be taken 8 supporting springs, the other conditions remaining unchanged?

3. What magnitude must the spring constant in problem 1 have in order to have the maximum disturbing force transmitted to the foundation equal to one-tenth of the centrifugal force $P\omega^2$?

6. Other Technical Applications.—*Oscillator.*—For determining the frequency of free vibrations of structures a special device called the *Oscillator* * is sometimes used. It consists of two discs rotating in a vertical plane with constant speed in opposite directions, as shown in Fig. 19. The bearings of the discs are housed in a rigid frame which must be rigidly attached to the structure, the vibrations of which are studied. By attaching to the discs the unbalanced weights symmetrically situated with respect to vertical axis mn, the centrifugal forces $P\omega^2$ which are produced during rotation of the discs have a resultant $2P\omega^2 \sin \omega t$ acting along the axis mn.† Such a pulsating force produces forced vibrations of the

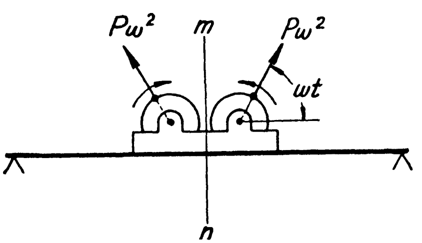

FIG. 19.

* Such an oscillator is described in a paper by W. Späth, see V.D.I. vol. 73, 1929.

† It is assumed that the effect of vibrations on the inertia forces of the unbalanced weights can be neglected.

structure which can be recorded by a vibrograph. By gradually changing the speed of the discs the number of revolutions per second at which the amplitude of forced vibrations of the structure becomes a maximum can be established. Assuming that this occurs at resonance,* the frequency of free vibration of the structure is equal to the above found number of revolutions per second of the discs.

Frahm's Vibration Tachometer.†—An instrument widely used for measuring the frequency of vibrations is known as Frahm's tachometer. This consists of a system of steel strips built in at their lower ends as shown in Fig. 20. To the upper ends of the strips small masses are attached, the magnitudes of which are adjusted in such a manner that the system

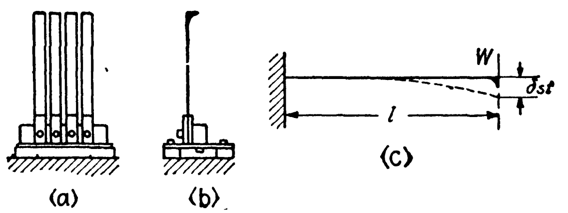

FIG. 20

of strips represents a definite series of frequencies. The difference between the frequencies of any two consecutive strips is usually equal to half a vibration per second.

In figuring the frequency a strip can be considered as a cantilever beam (Fig. 20–*c*). In order to take into consideration the effect of the mass of the strip on the vibration it is necessary to imagine that one quarter of the weight W_1 of the strip is added ‡ to the weight W, the latter being concentrated at the end. Then,

$$\delta_{st} = \frac{(W + W_1/4)l^3}{3EI}.$$

This statical deflection must be substituted in eq. 6 in order to obtain the period of natural vibration of the strip. In service the instrument is attached to the machine, the frequency vibrations of which is to be

* For a more accurate discussion of this question the effect of damping must be considered (see Art. 9).

† This instrument is described by F. Lux, E. T. Z., 1905, pp. 264-387.

‡ A more detailed consideration of the effect of the mass of the beam on the period of vibration is given in article 16.

measured. The strip whose period of natural vibration is nearest to the period of one revolution of the machine will be in a condition near resonance and a heavy vibration of this strip will be built up. From the frequency of the strip, which is known, the speed of the machine can be obtained.

Instead of a series of strips of different lengths and having different masses at the ends, one strip can be used having an adjustable length. The frequency of vibration of the machine can then be found by adjusting the length of the strip in this instrument so as to obtain resonance. On this latter principle the well known Fullarton vibrometer is built (see p. 443).

Indicator of Steam Engines.—Steam engine indicators are used for measuring the variation of steam pressure in the engine cylinder. The accuracy of the records of such indicators will depend on the ability of the indicator system, consisting of piston, spring and pencil, to follow exactly the variation of the steam pressure. From the general discussion of the article 3 it is known that this condition will be satisfied if the frequency of free vibrations of the indicator system is very high in comparison with that of the steam pressure variation in the cylinder.

Let A = .20 sq. in. is area of the indicator piston,
W = .133 lb. is weight of the piston, piston rod and reduced weight of other parts connected with the piston,
s = .1 in. displacement of the pencil produced by the pressure of one atmosphere (15 lbs. per sq. in.),
n = 4 is the ratio of the displacement of the pencil to that of the piston.

From the condition that the pressure on the piston equal to $15 \times .2 = 3.00$ lbs. produces a compression of the spring equal to $\frac{1}{4} \times .1 = .025$ in., we find that the spring constant is:

$$k = 3.00 : .025 = 120 \text{ lbs. in}^{-1}.$$

The frequency of the free vibrations of the indicator is (see eq. (6))

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_{st}}} = \frac{1}{2\pi}\sqrt{\frac{gk}{W}} = \frac{1}{2\pi}\sqrt{\frac{386.120}{.133}} = 94 \text{ per sec.}$$

This frequency can be considered as sufficiently high in comparison with the usual frequency of steam engines and the indicator's record of steam pressure will be sufficiently accurate. In the case of high speed engines, however, such an instrument may give completely unreliable records * under certain conditions.

Locomotive Wheel Pressure on the Rail.—It is well known that inertia forces of counter weights in locomotive wheels produce additional pressure on the track. This effect of counterweights can easily be obtained by using the theory of forced vibrations. Let W is the weight of the wheel and of all parts rigidly connected to the wheel, Q is spring borne weight, P is centrifugal force due to unbalance, ω is angular velocity of the wheel. Considering then the problem as one of statics, the vertical pressure of the wheel on the rail, Fig. 21, will be equal to

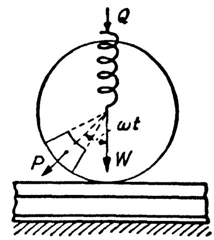

FIG. 21.

$$Q + W + P \cos \omega t. \tag{a}$$

At slow speed this expression represents a good approximation for the wheel pressure. In order to get this pressure with greater accuracy, forced vibrations of the wheel on the rail produced by the periodical vertical force $P \cos \omega t$ must be considered. Let k denote the vertical load on the rail necessary to produce the deflection of the rail equal to unity directly under the load and δ_{st}, the deflection produced by the weight W, then,

$$\delta_{st} = \frac{W}{k}.$$

The period of free vibrations of the wheel on the rail is given by the equation † (see eq. (5)).

$$\tau = 2\pi \sqrt{\frac{W}{kg}}. \tag{b}$$

The period of one revolution of the wheel, i.e., the period of the disturbing force $P \cos \omega t$, is

$$\tau_1 = \frac{2\pi}{\omega}.$$

* The description of an indicator for high frequency engines (Collins Micro-Indicator) is given in Engineering, Vol. 113, p. 716 (1922). Symposium of Papers on Indicators, see Proc. Meetings of the Inst. Mech., Eng., London, Jan. (1923).

† In this calculation the mass of the rail is neglected and the compressive force Q in the spring is considered as constant. This latter assumption is justified by the fact that the period of vibration of the engine cab on its spring is usually very large in comparison with the period of vibration of the wheel on the rail, therefore vibrations ot tne wheel will not be transmitted to the cab and variations in the compression of the spring will be very small (see Art. 4).

Now, by using eq. (20), it can be concluded that the dynamical deflection of the rail produced by the force P will be larger than the corresponding statical deflection in the ratio,

$$\frac{1}{1 - \left(\frac{\tau}{\tau_1}\right)^2}. \tag{c}$$

The pressure on the rail produced by the centrifugal force P will also increase in the same ratio and the maximum wheel pressure will be given by

$$Q + W + \frac{P}{1 - \left(\frac{\tau}{\tau_1}\right)^2}. \tag{d}$$

For a 100 lb. rail, a modulus of the elastic foundation equal to 1500 lbs. per sq. in. and $W = 6000$ lbs. we will have *

$$\tau = .068 \text{ sec.}$$

Assuming that the wheel performs five revolutions per sec. we obtain

$$\tau_1 = .2 \text{ sec.}$$

Substituting the values of τ and τ_1 in the expression (c) it can be concluded that the dynamical effect of the counterbalance will be about 11% larger than that calculated statically.

7. Damping.—In the previous discussion of free and forced vibrations it was assumed that there are no resisting forces acting on the vibrating body. As a result of this assumption it was found that in the case of free vibrations the amplitude of vibrations remains constant, while experience shows that the amplitude diminishes with the time, and vibrations are gradually damped out. In the case of forced vibrations at resonance it was found that the amplitude of vibration can be indefinitely built up, but, as we know, due to damping, there is always a certain upper limit below which the amplitude always remains. To bring an analytical discussion of vibration problems in better agreement with actual conditions *damping forces* must be taken into consideration. These damping forces may arise from several different sources such as friction between the dry sliding surfaces of the bodies, friction between lubricated surfaces, air or fluid resistance, electric damping, internal friction due to imperfect elasticity of vibrating bodies, etc.

* See S. Timoshenko and J. M. Lessells, "Applied Elasticity," p. 334 (1925).

In the case of friction between dry surfaces the Coulomb-Morin law is usually applied.* It is assumed that in the case of dry surfaces the friction force F is proportional to the normal component N of the pressure acting between the surfaces, so that

$$F = \mu N, \tag{a}$$

where μ is the *coefficient of friction* the magnitude of which depends on the materials of the bodies in contact and on the roughness of their surfaces.

Experiments show that the force F required to overcome friction and start a motion is larger than the force necessary to maintain a uniform motion. Thus usually larger values are assumed for the coefficients of friction at rest than for the coefficients of friction during motion. It is usually assumed also that the coefficient of friction during motion is independent of the velocity so that Coulomb's law can be represented by a line BC, parallel to abscissa axis, as shown in Fig. 22. By the position of the point A in the same figure the coefficient of friction at rest is given. This law agrees satisfactorily with experiments in the case of smooth surfaces. When the surfaces are rough the coefficient of friction depends on velocity and diminishes with the increase of the velocity as shown in Fig. 22 by the curve AD.†

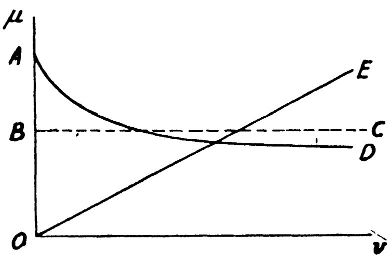

FIG. 22.

In the case of friction between lubricated surfaces the friction force does not depend on materials of the bodies in contact but on the *viscosity* of lubricant and on the velocity of motion. In the case of perfectly lubricated surfaces in which there exists a continuous lubricating film between the sliding surfaces it can be assumed that friction forces are proportional both to the viscosity of the lubricant and to the velocity. The coefficient of friction, as a function of velocity, is represented for this case, in Fig. 22, by the straight line OE.

* C. A. Coulomb, Mémoires de Math. et de Phys., Paris 1785; see also his "Theorie des machines simples," Paris, 1821. A. Morin, Mémoires prés. p. div. sav., vol. 4, Paris 1833 and vol. 6, Paris, 1935. For a review of the literature on friction, see R. v. Mises, Encyklopädie d Math. Wissenschaften, vol. 4, p. 153. For references to new literature on the same subject see G. Sachs, Z. f. angew. Math. und Mech., Vol. 4, p. 1, 1924; H. Fromm, Z. f. angew. Math. und Mech. Vol. 7, p. 27, 1927 and Handbuch d. Physik. u. Techn. Mech. Vol. 1, p. 751, 1929.

† The coefficient of friction between the locomotive wheel and the rail were investigated by Douglas Galton, See "Engineering," vol. 25 and 26, 1878 and vol. 27, 1879.

We obtain also resisting forces proportional to the velocity if a body is moving in a viscous fluid with a small velocity or if a moving body causes fluid to be forced through narrow passages as in the case of dash pots.* In further discussion of all cases in which friction forces are proportional to velocity we will call these forces *viscous damping.*

In the case of motion of bodies in air or in liquid with larger velocities a resistance proportional to the square of velocity can be assumed with sufficient accuracy.

The problems of vibration in which damping forces are not proportional to the velocity can be discussed in many cases with sufficient accuracy by replacing actual resisting forces by an *equivalent viscous damping* which is determined in such a manner as to produce same dissipation of energy per cycle as that produced by the actual resisting forces. In this manner, the damping due to internal friction can be treated. For this purpose it is necessary to know for the material of a vibrating body the amount of energy dissipated per cycle as a function of the maximum stress. This can be determined by measuring the *hysteresis loop* obtained during deformation.† Several simple examples of vibrations with damping will now be considered.

8. Free Vibration with Viscous Damping.—Consider again the vibration of the system shown in Fig. 1 and assume that the vibrating body W encounters in its motion a resistance proportional to the velocity. In such case, instead of equation of motion (1), we obtain

$$\frac{W}{g}\ddot{x} = W - (W + kx) - c\dot{x}. \tag{a}$$

The last term on the right side of this equation represents the damping force, proportional to velocity $\dot{x}$. The minus sign shows that the force is acting in the direction opposite to the velocity. The coefficient c is a constant depending on the kind of the damping device and numerically is equal to the magnitude of the damping force when the velocity is equal to unity. Dividing equation (a) by W/g and using notations

$$p^2 = kg/W \quad \text{and} \quad cg/W = 2n, \tag{25}$$

* See experiments by A. Stodola, Schweiz. Banzeitung, vol. 23, p. 113, 1893.

† Internal friction is a very important factor in the case of torsional vibrations of shafts and a considerable amount of experimental data on this subject have been obtained during recent years. See O. Föppl, V.D.I. vol. 74, p. 1391, 1930; Dr. Dorey's paper read before Institution of Mechanical Engineers, November, 1932; I. Geiger, V.D.I. vol. 78, p. 1353, 1934.

we obtain for free vibrations with viscous damping the following equation

$$\ddot{x} + 2n\dot{x} + p^2x = 0. \tag{26}$$

In discussing this equation we apply the usual method of solving linear differential equations with constant coefficients, and assume a solution of it in the form

$$x = e^{rt}, \tag{b}$$

in which e is the base of natural logarithms, t is time and r is a constant which must be determined from the condition that expression (b) satisfies equation (26). Substituting (b) in eq. (26) we obtain

$$r^2 + 2nr + p^2 = 0,$$

from which

$$r = - n \pm \sqrt{n^2 - p^2}. \tag{c}$$

Let us consider first the case when the quantity n^2, depending on damping, is smaller than the quantity p^2. In such case the quantity

$$p_1{}^2 = p^2 - n^2 \tag{27}$$

is positive and we get for r two complex roots:

$$r_1 = - n + p_1i \quad \text{and} \quad r_2 = - n - p_1i.$$

Substituting these roots in expression (b) we obtain two particular solutions of the equation (26). The sum or the difference of these two solutions multiplied by any constant will be also a solution. In this manner we get solutions

$$x_1 = \frac{C_1}{2}(e^{r_1t} + e^{r_2t}) = C_1e^{-nt}\cos p_1t,$$

$$x_2 = \frac{C_2}{2i}(e^{r_1t} - e^{r_2t}) = C_2e^{-nt}\sin p_1t.$$

Adding them together the general solution of eq. 26 is obtained in the following form

$$x = e^{-nt}(C_1 \cos p_1t + C_2 \sin p_1t), \tag{28}$$

in which C_1 and C_2 are constants which in each particular case must be determined from the initial conditions.

The expression in parenthesis of solution (28) is of the same form as we had before for vibrations without damping (see expression 4). It represents a periodic function with the period

$$\tau = \frac{2\pi}{p_1} = \frac{2\pi}{p}\frac{1}{\sqrt{1 - n^2/p^2}}. \tag{29}$$

Comparing this with the period $2\pi/p$, obtained before for vibrations without damping, we see that due to damping the period of vibration increases, but if n is small in comparison with p, this increase is a small quantity of second order. Therefore, in practical problems, it can be assumed with sufficient accuracy that a small viscous damping does not affect the period of vibration.

The factor e^{-nt} in solution (28) gradually decreases with the time and the vibrations, originally generated, will be gradually damped out.

To determine the constants C_1 and C_2 in solution (28) let us assume that at the initial instant $t = 0$ the vibrating body is displaced from its position of equilibrium by the amount x_0 and has an initial velocity $\dot{x}_0$. Substituting $t = 0$ in expression (28) we then obtain

$$x_0 = C_1. \tag{d}$$

Differentiating the same expression with respect to time and equating it to $\dot{x}_0$, for $t = 0$, we obtain

$$C_2 = (\dot{x}_0 + nx_0)/p_1. \tag{e}$$

Substituting (*d*) and (*e*) into solution (28) we obtain

$$x = e^{-nt}\left(x_0 \cos p_1 t + \frac{\dot{x}_0 + nx_0}{p_1}\sin p_1 t\right). \tag{30}$$

The first term in this expression proportional to cos $p_1 t$, depends only on the initial displacement x_0 and the second term, proportional to sin $p_1 t$ depends on both, initial displacement x_0 and initial velocity $\dot{x}_0$. Each term can be readily represented by a curve. The wavy curve in Fig. 23 represents the first term. This curve is tangent to the curve $x = x_0 e^{-nt}$ at the points m_1, m_2, m_3, where $t = 0, t = \tau, t = 2\tau, \ldots$; and to the curve $x = -x_0 e^{-nt}$ at the points m_1', m_2', ... where $t = \tau/2, t = 3\tau/2, \ldots$ These points do not coincide with the points of extreme displacements of the body from the position of equilibrium and it is easy to see that due to damping, the time interval necessary for displacement of the body from a middle position to the subsequent extreme position is less than that necessary to return from an extreme position to the subsequent middle position.

The rate of damping depends on the magnitude of the constant n (see eq. (25)). It is seen from the general solution (30) that the amplitude of the vibration diminishes after every cycle in the ratio

$$e^{-n\tau} : 1, \tag{f}$$

i.e., it decreases following the law of geometrical progression. Equation (f) can be used for an experimental determination of the coefficient of damping n. It is only necessary to determine by experiment in what

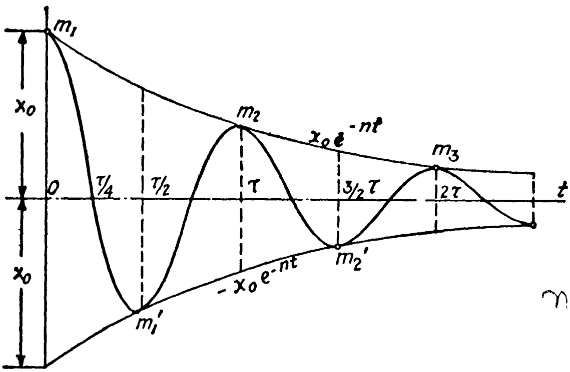

FIG. 23.

proportion the amplitude of vibration is diminished after a given number of cycles.

The quantity

$$n\tau = \frac{2\pi}{p}\frac{n}{\sqrt{1 - n^2/p^2}}, \tag{31}$$

on which the rate of damping depends, is usually called the *logarithmic decrement*. It is equal to the difference between the logarithms of the two consecutive amplitudes measured at the instants t and $t + \tau$.

In discussing vibrations without damping the use of a rotating vector for representing motion was shown. Such vector can be used also in the case of vibrations with damping. Imagine a vector $\overline{OA}$, Fig. 24, of variable magnitude x_0e^{-nt} rotating with a constant angular velocity p_1. Measuring the angle of rotation in the counter clockwise direction from the x-axis, the projection OA_1 of the vector is equal to $x_0e^{-nt} \cos p_1t$ and represents the first term of the expression (30). In the same manner, by taking a vector $\overline{OB}$ equal to $e^{-nt}(\dot{x}_0 + nx_0)/p_1$ and perpendicular to $\overline{OA}$ and projecting it on the axis, we get the second term of solution (30). The total expression (30) will be obtained by projecting on the x-axis the vector $\overline{OC}$ which is the geometrical sum of the vectors $\overline{OA}$ and $\overline{OB}$. The magnitude of this vector is

$$\overline{OC} = \sqrt{\overline{OA}^2 + \overline{OB}^2} = e^{-nt} \sqrt{x_0^2 + (\dot{x}_0 + nx_0)^2/p_1^2}, \tag{g}$$

and the angle which it makes with x-axis is $p_1t - \alpha$ where

$$\alpha = \text{arc tan} \frac{\dot{x}_0 + nx_0}{p_1x_0}. \tag{h}$$

From this discussion it follows that expression (30) can be put in the following form

$$x = e^{-nt} \sqrt{x_0^2 + (\dot{x}_0 + nx_0)^2/p_1^2} \cos (p_1t - \alpha). \tag{30'}$$

During rotation of the vector $\overline{OC}$, in Fig. 24, the point C describes a logarithmic spiral the tangent to which makes a constant angle equal to

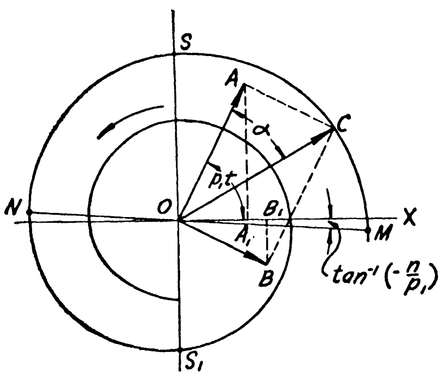

FIG. 24.

arc tan $(-n/p_1)$ with the perpendicular to the radius vector $\overline{OC}$. The extreme positions of the vibrating body correspond to the points at which the spiral has vertical tangents. These points are defined by the intersections of the spiral with the straight line MN, Fig. 24. The points of intersection of the spiral with the vertical axis define the instants when the vibrating body is passing through the equilibrium position. It is clearly seen that the time interval required for the displacement of the body from the equilibrium position to the extreme position, say the time given by the angle SON, in Fig. 24, is less than that necessary to return from the extreme position to the subsequent equilibrium position, as given by the angle NOS_1. But the time between the two consecutive extreme positions of the body, such as given by the points M and N in Fig. 24 is always the same and equal to half of the period τ.

In the foregoing discussion of equation (26) we assumed that $p^2 > n^2$. If $p^2 < n^2$ both roots (c) become real and are negative. Substituting them in expression (b) we obtain the two particular solutions of equation (26) and the general solution of the same equation becomes

$$x = C_1 e^{r_1 t} + C_2 e^{r_2 t}. \tag{k}$$

The solution does not contain any longer a periodical factor and does not represent a vibratory motion. The viscous resistance is so large that the body, displaced from its equilibrium position does not vibrate and only creeps gradually back to that position.

The *critical value of* damping at which the motion loses its vibratory character is given by the condition $n = p$, and by using notations (25) we obtain for this case:

$$c_{cr} = 2\sqrt{\frac{kW}{g}}. \tag{l}$$

PROBLEMS

1. A body vibrating with viscous damping (Fig. 1) makes ten complete oscillations per second. Determine n in eq. (26) if after an elapse of 10 seconds the amplitude of vibration is reduced to 0.9 of the initial. Determine in what proportion the period of vibration decreases if damping is removed. Calculate the logarithmic decrement.

Solution. Assuming that motion is given by equation

$$x = x_0 e^{-nt} \cos p_1 t$$

and substituting in this equation $x = 0.9x_0$, $t = 10$, $p_1 = 20\pi$ we obtain

$$e^{10n} = \frac{1}{.9} = 1.111,$$

from which $n = .01054$.

The effect of damping on the period of vibration is given, in eq. (29), by factor $1/\sqrt{1 - n^2/p^2} = p/\sqrt{p^2 - n^2} = p/p_1$. Substituting $p = \sqrt{p_1^2 + n^2} = p_1\sqrt{1 + n^2/p_1^2}$ we see that by removing damping the period of vibration decreases in the ratio $1/\sqrt{1 + n^2/p_1^2} \approx 1 - \frac{1}{2}\frac{n^2}{p_1^2}$, in which n and p_1 have the values calculated above. The logarithmical decrement is $n\tau = .01054 \cdot 0.1 = .001054$.

2. To the body weighing 10 lb. and suspended on the spring, Fig. 1, a dash pot mechanism is attached which produces a resistance of .01 lb. at a velocity 1 in. per sec. In what ratio is the amplitude of vibration reduced after ten cycles if the spring constant is 10 lb. per in.

Solution. After 10 cycles the amplitude of oscillation reduces in the ratio $1/e^{10n\tau}$. Substituting, from (25) and (29),

$$n = \frac{cg}{2W}, \quad n\tau = 2\pi\sqrt{\frac{c^2g}{4kW}} \bigg/ \sqrt{1 - c^2g/4kW} \approx .0617,$$

the ratio becomes $1/e^{.617} = .539$.

9. Forced Vibrations with Viscous Damping.—In discussing forced vibration with viscous damping we assume that in addition to forces considered in the previous article a disturbing force $P \sin \omega t$ is acting on the vibrating body, Fig. 1. Then instead of equation (*a*) of the previous article, we obtain

$$\frac{W}{g}\ddot{x} = W - (W + kx) - c\dot{x} + P \sin \omega t.$$

By using notations (25) this equation becomes

$$\ddot{x} + 2n\dot{x} + p^2x = \frac{Pg}{W}\sin \omega t. \tag{32}$$

The general solution of this equation is obtained by adding to the solution of the corresponding homogeneous equation (26), p. 33, a particular solution of equation (32). This later solution will have the form

$$x_1 = M \sin \omega t + N \cos \omega t, \tag{a}$$

in which M and N are constants. Substituting this expression into equation (32) we find that it is satisfied if the constants M and N satisfy the following linear equations

$$-N\omega^2 + 2M\omega n + Np^2 = 0,$$

$$-M\omega^2 - 2N\omega n + Mp^2 = \frac{Pg}{W},$$

from which

$$M = \frac{Pg}{W} \cdot \frac{p^2 - \omega^2}{(p^2 - \omega^2)^2 + 4n^2\omega^2}\ ; \quad N = -\frac{Pg}{W}\frac{2n\omega}{(p^2 - \omega^2)^2 + 4n^2\omega^2}. \tag{b}$$

Substituting these expressions in (*a*) we obtain the required particular solution. Adding it to the general solution (28) of the homogeneous equation the general solution of equation (32) becomes

$$x = e^{-nt}(C_1 \cos p_1t + C_2 \sin p_1t) + M \sin \omega t + N \cos \omega t. \tag{c}$$

The first member on the right side, having the factor e^{-nt}, represents the free damped vibration discussed in the previous article. The two other terms, having the same frequency as the disturbing force, represent *forced vibration.*

The expression for the forced vibration can be simplified by using rotating vectors as before, see p. 35. Take a vector $\overline{OD}$ of magnitude M rotating with a constant angular velocity ω in the counter clockwise direction. Then measuring angles as shown in Fig. 25, the projection of this vector on the x-axis gives us the first term of expression (a) for the forced vibration. The second term of the same expression is obtained by taking the projection on the x-axis of the vector $\overline{OB}$ perpendicular to $\overline{OD}$ the magnitude of which is equal to the absolute value of N and which is directed so as to take care of the negative sign of N in the second of expressions (b). The algebraical sum of the projections of the two vectors $\overline{OD}$ and $\overline{OB}$ can be replaced by the projection of their geometrical sum represented by the vector $\overline{OC}$. The magnitude of this vector, which we denote by A, is obtained from the triangle ODC and, by using expressions (b), is

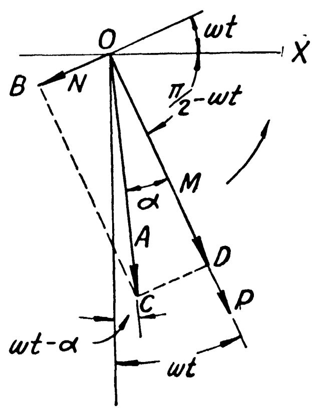

FIG. 25.

$$A = \sqrt{M^2 + N^2} = \frac{Pg}{W}\frac{1}{\sqrt{(p^2 - \omega^2)^2 + 4n^2\omega^2}},$$

from which, by taking p^2 out of the radical and substituting for it its value from (25), we obtain

$$A = \frac{P}{k}\cdot\frac{1}{\sqrt{\left(1 - \frac{\omega^2}{p^2}\right)^2 + \frac{4n^2\omega^2}{p^4}}} = \delta_{st}\cdot\frac{1}{\sqrt{\left(1 - \frac{\omega^2}{p^2}\right)^2 + \frac{4n^2\omega^2}{p^4}}}, \tag{33}$$

in which δ_{st} denotes the deflection of the spring, in Fig. 1, when a vertical force P is acting statically. The angle α between the vectors $\overline{OD}$ and $\overline{OC}$ is determined from the equation

$$\tan\alpha = \frac{-N}{M} = \frac{2n\omega}{p^2 - \omega^2}. \tag{34}$$

Projecting now vector $\overline{OC}$ on the x-axis we obtain the following expression for the forced vibration

$$x_1 = \delta_{st} \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{p^2}\right)^2 + \frac{4n^2\omega^2}{p^4}}} \sin(\omega t - \alpha). \tag{35}$$

It is seen that the amplitude of the forced vibration is obtained by multiplying the statical deflection δ_{st} by the absolute value of the factor

$$1/\sqrt{\left(1 - \frac{\omega^2}{p^2}\right)^2 + \frac{4n^2\omega^2}{p^4}},$$

which is called the *magnification factor.* The magnitude of it depends on the ratio ω/p of the circular frequencies of the disturbing force and of the free vibration without damping, and also on the ratio n/p which, in most practical cases, is a small quantity. By taking this later ratio equal to zero we obtain for the amplitude of forced vibration the value found before in discussing vibrations without damping, see eq. (20) p. 15.

In Fig. 26 the values of the magnification factor for various values of the ratio $2n/p$ are plotted against the values of ω/p. From this figure it is seen that in the cases when the frequency of the disturbing force is small in comparison with that of free vibration of the system, the magnification factor approaches the value of unity, hence the amplitude of forced vibration is approximately equal to δ_{st}. This means that in such cases the deflection of the spring at any instant can be calculated with sufficient accuracy by assuming that the disturbing force $P \sin \omega t$ is acting statically.

We have another extreme case when ω is large in comparison with p, i.e., when the frequency of the disturbing force is large in comparison with the frequency of free vibration of the system. In such a case the magnification factor becomes very small and the amplitude of forced vibration is small also.

The curves shown in Fig. 26 are very close together for both extreme cases mentioned above. This indicates that for these cases the effect of damping is of no practical importance in calculating the amplitudes of forced vibrations and the amplitude calculated before by neglecting damping, see Art. (3), can be used with sufficient accuracy.

When the frequency of the disturbing force approaches the frequency of the free vibration of the system the magnification factor increases rapidly and, as we see from the figure, its value is very sensitive to changes in the magnitude of damping especially when this damping is small. It is seen also that the maximum values of the magnification factor occur at values of the ratio ω/p which are somewhat smaller than unity. By equating to zero the derivative of the magnification factor with respect to ω/p it can be shown that this maximum occurs when

$$\frac{\omega^2}{p^2} = 1 - \frac{2n^2}{p^2}. \tag{d}$$

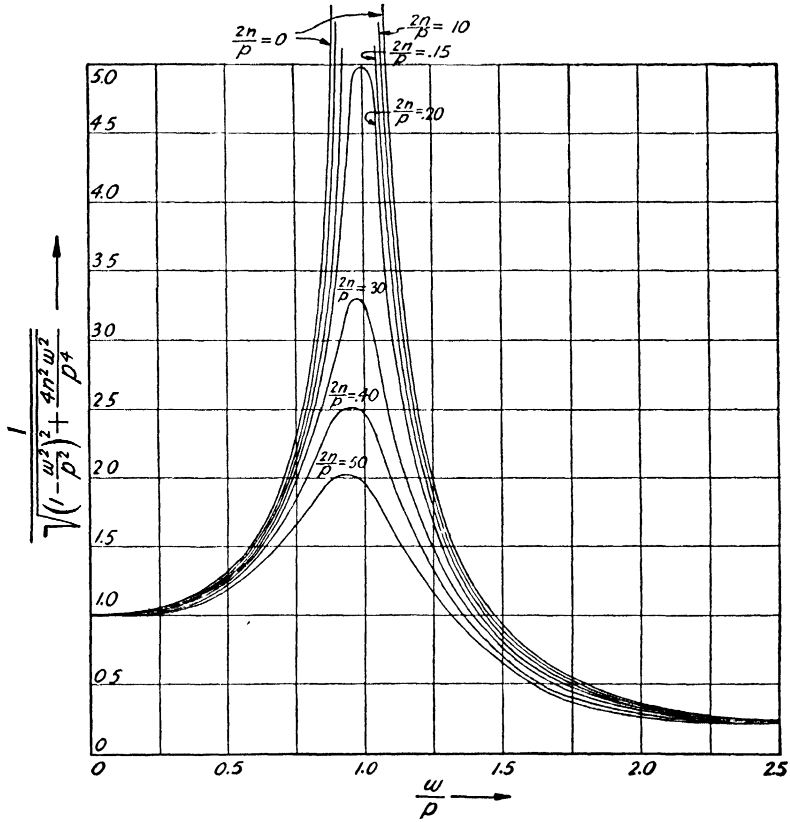

FIG. 26.

Since n is usually very small in comparison with p the values of the frequency ω at which the amplitude of forced vibration becomes a maximum differ only very little from the frequency p of the free vibration of the system without damping and it is usual practice to take, in calculating maximum amplitudes, $\omega = p$, in which case, from eq. (33),

$$A_{\max} = \frac{\delta_{st} p}{2n}. \tag{36}$$

We have discussed thus far the magnitude of the amplitude of forced vibration given in Fig. 25 by the magnitude of the vector $\overline{OC}$. Let us consider now the significance of the angle α defining the direction of the vector $\overline{OC}$. For this purpose we use a rotating vector for representation of the disturbing force. Since this force is proportional to sin ωt the vector $\overline{OP}$, representing the force, coincides in Fig. 25 with the direction of the vector $\overline{OD}$, and its projection on the x-axis gives at any instant the magnitude of the disturbing force. Due to the angle α between the vectors $\overline{OP}$ and $\overline{OC}$ the forced vibration always lags behind the disturbing force.

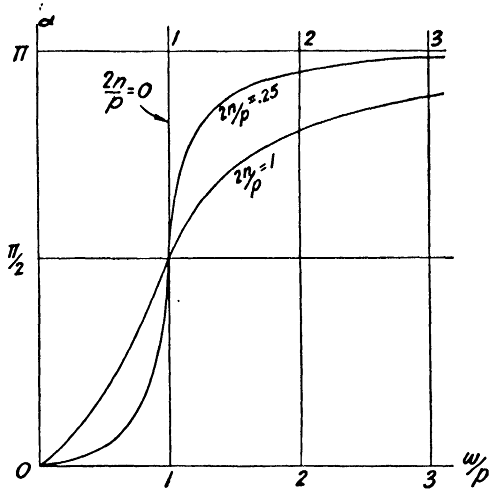

Fig. 27.

When the vector $\overline{OP}$ coincides with the x-axis and the disturbing force is maximum the displacement of the body, given by the projection of $\overline{OC}$ on the x-axis, has not yet reached its maximum value and becomes a maximum only after an interval of time equal to α/ω when $\overline{OC}$ coincides with the x-axis. The angle α represents the *phase difference* between the disturbing force and the forced vibration. From equation (34) we see that when $\omega < p$, i.e., when the frequency of the disturbing force is less than the frequency of the natural undamped vibration, tan α is positive and α is less than $\pi/2$. For $\omega > p$, tan α is negative and $\alpha > \pi/2$. When $\omega = p$, tan α becomes infinite and the difference in phase α becomes equal to $\pi/2$. This means that during such motion the vibrating body passes through the middle position at the instant when the disturbing force attains its maximum value. In Fig. 27 the values of α are plotted against the values of the ratio ω/p for various values of damping. It is seen that in the region of resonance ($\omega = p$) a very sharp variation in the phase difference α takes place when damping is small. Under the limiting condition when $n = 0$ an abrupt change in the phase difference from $\alpha = 0$ to $\alpha = \pi$ occurs at resonance and instead of a curve we obtain in Fig. 27 a broken line 0113. This later condition corresponds to the case of undamped forced vibration discussed before, see p. 16.

When the expression (35) for the forced vibration is obtained the force in the spring, the damping force and the inertia force of the vibrating body, Fig. 1, can be readily calculated for any instant. Taking, from (33) and (35),

$$x_1 = A \sin (\omega t - \alpha), \tag{e}$$

we obtain the force in the spring, due to the displacement from the equilibrium position, equal to

$$-kx_1 = -kA \sin (\omega t - \alpha). \tag{f}$$

The damping force, proportional to velocity, is

$$-c\dot{x}_1 = -cA\omega \cos (\omega t - \alpha), \tag{g}$$

and the inertia force of the vibrating body is

$$-\frac{W}{g}\ddot{x}_1 = \frac{W}{g}A\omega^2 \sin (\omega t - \alpha). \tag{h}$$

All these forces together with the disturbing force $P \sin \omega t$ can be obtained by projecting on the x-axis the four vectors the magnitudes and directions of which are shown in Fig. 28. From d'Alembert's principle it follows that the sum of all these forces is zero, hence

$$P \sin \omega t - kx_1 - c\dot{x}_1 - \frac{W}{g}\ddot{x}_1 = 0, \tag{k}$$

which is the same equation as equation (32). This equation holds for any value of the angle ωt, hence the geometrical sum of the four vectors, shown in Fig. 28, is zero and the sum of their projections on any axis must be zero. Making projections on the directions Om and On we obtain

$$\frac{W}{g}A\omega^2 + P \cos \alpha - kA = 0,$$
$$-cA\omega + P \sin \alpha = 0.$$

From these equations A and α can be readily calculated and the formulae (33) and (34) for the amplitude of forced vibration and for the phase difference can be obtained.

Figure 28 can be used in discussing how the phase angle α and the amplitude A vary with the frequency ω of the disturbing force. When ω is small the damping force is also small. The direction of the force P must be very close to the direction Om and since the inertia force proportional to ω^2 in this case is very small the force P must be approximately equal to

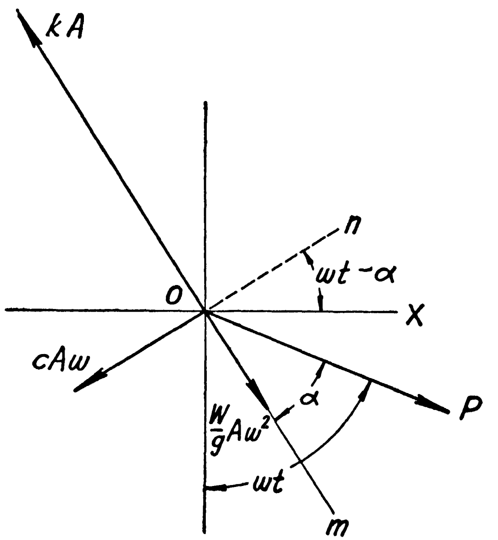

Fig. 28.

the spring force kA; thus the amplitude of vibration must be close to the statical deflection δ_{st}. With a growing value of ω the damping force increases and the phase angle α increases to the magnitude at which the component of the force P in the direction On balances the damping forces. At the same time the inertia force increases as ω^2 and to balance this force together with the component of P in the Om direction a larger spring force, i.e., a larger amplitude A is required. At resonance ($\omega = p$) the inertia force balances the spring force and the force P acting in the direction On, balances the damping force. Thus the phase angle becomes equal to $\pi/2$. With further growing of ω the angle α becomes larger than $\pi/2$ and the component of the force P is added to the force kA of the spring so that the inertia force can be balanced at a smaller value of the amplitude. Finally, at very large values of ω, the angle α approaches π, the force P acts approximately in the direction of the spring force kA. The amplitude A, the damping force and the spring force become small and the force P balances the inertia force.

Let us consider now the work per cycle produced by the disturbing force during steady forced vibration.* The force acting at any instant is $P \sin \omega t$ and the velocity of its point of application is $\dot{x}_1 = A\omega \cos (\omega t - \alpha)$, hence the work produced in an infinitely small interval of time is

$$P \sin \omega t A\omega \cos (\omega t - \alpha) dt,$$

and the work per cycle will be

$$\int_0^\tau P \sin \omega t A\omega \cos (\omega t - \alpha) dt = \frac{A\omega P}{2} \int_0^\tau [\sin (2\omega t - \alpha) + \sin \alpha] dt$$
$$= \frac{A\omega P\tau \sin \alpha}{2} = \pi A P \sin \alpha. \tag{37}$$

This work must be equal to the energy dissipated during one cycle due to damping force. The magnitude of this force is given by expression (g). Multiplying it by $\dot{x}_1 dt$ and integrating in the interval from 0 to τ we get for the energy dissipated per cycle the expression

$$\int_0^\tau cA^2\omega^2 \cos^2(\omega t - \alpha) dt = \frac{cA^2\omega^2\tau}{2} = \pi c A^2 \omega. \tag{38}$$

Thus the energy dissipated per cycle increases as the square of the amplitude.

Expressions (37) and (38) can be used for calculating the maximum amplitude which a given disturbing force may produce when damping is known. It may be assumed with sufficient accuracy that this amplitude occurs at resonance, when $\omega = p$ and $\alpha = \pi/2$. Substituting $\sin \alpha = 1$ in eq. (37) and equating the work done by the disturbing force to the energy dissipated we obtain

$$\pi A P = \pi c A^2 \omega,$$

from which

$$A_{\max} = \frac{P}{c\omega}. \tag{39}$$

* Due to presence of the factor e^{-nt} in the first term on the right side of eq. (c) (see p. 38) the free vibrations will be gradually damped out and steady forced vibrations will be established.

This expression can be easily brought in coincidence with the expression (36) by using notations (25).

From Fig. 25 it is seen that the quantity $A \sin \alpha$ is equal to the absolute value of N given by expression (b). Substituting this value into formula (37) we obtain for the work per cycle of the disturbing force the following expression

$$\frac{\pi P^2 g}{W} \frac{2n\omega}{(p^2 - \omega^2)^2 + 4n^2\omega^2} = \frac{2\pi}{\omega} \frac{P^2 g}{W} \frac{2n/p}{2p[(p/\omega - \omega/p)^2 + (2n/p)^2]}.$$

Using notations

$$2n/p = \gamma \qquad p/\omega = 1 + z \tag{1}$$

we represent this work in the following form

$$\frac{2\pi}{\omega} \frac{P^2 g}{W} \frac{\gamma}{2p\left[\left(1 + z - \dfrac{1}{1+z}\right)^2 + \gamma^2\right]},$$

and since $2\pi/\omega$ is the period of vibration the *average work* per second is

$$\frac{P^2 g}{W} \frac{\gamma}{2p\left[\left(1 + z - \dfrac{1}{1+z}\right)^2 + \gamma^2\right]}. \tag{m}$$

Assuming that all quantities in this expression, except z, are given we conclude that the average work per second becomes maximum at resonance $(p = \omega)$ when z is zero.

In studying the variation of the average work per second near the point of resonance the quantity z can be considered as small and expression (m) can be replaced by the following approximate expression

$$\frac{P^2 g}{2pW} \cdot \frac{\gamma}{4z^2 + \gamma^2}.$$

The second factor of this expression is plotted against z in Fig. 29 for three different values of γ. It may be seen that with diminishing of damping the curves in the figure acquire a more and more pronounced peak at the resonance $(z = 0)$ and also that only near the resonance point the dissipated energy increases with decreasing damping. For points at a distance from resonance $(z = 0)$ the dissipated energy decreases with the decrease of damping.

In studying forced vibration with damping a geometrical representation in which the quantities $2n\omega$ and $p^2 - \omega^2$, entering in formulas (33) and (34), are considered as rectangular coordinates, is sometimes very useful. Taking

$$p^2 - \omega^2 = x \quad \text{and} \quad 2n\omega = y \tag{n}$$

and eliminating ω from these two equations we obtain the equation of a parabola:

$$x = p^2 - \frac{y^2}{4n^2}, \tag{o}$$

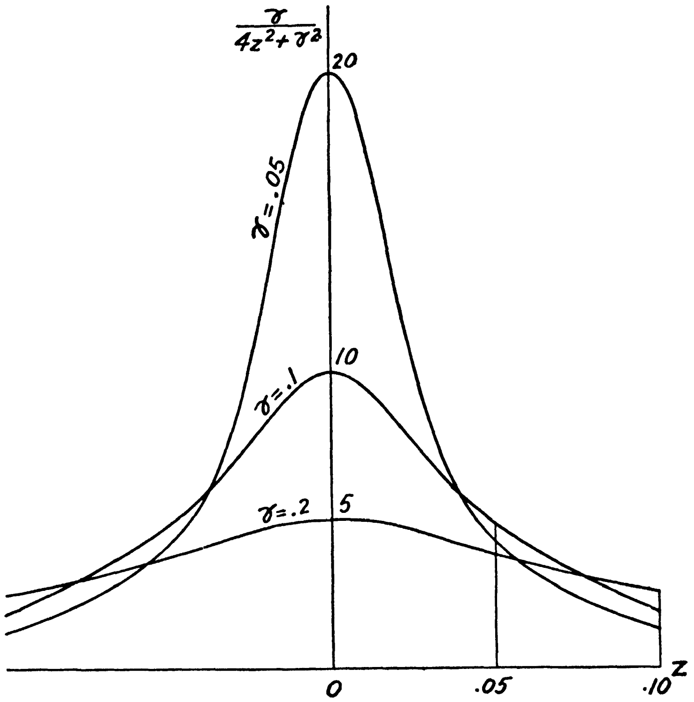

Fig. 29.

which is represented in Fig. 30. For $\omega = 0$, we have $y = 0$ and obtain the vertex of the parabola. For $\omega = p$, $x = 0$ and we obtain the intersection of the parabola with y-axis. For any given value of the frequency we readily obtain the corresponding point C on the parabola. Then, as seen from equations (33) and (34), the magnitude of the vector $\overline{OC}$ is inversely proportional to the amplitude of forced vibrations and the angle which it makes with x-axis is the phase angle α. For small damping n is small in comparison with p. Thus we obtain a very slender parabola and the shortest distance $\overline{OD}$ from the origin 0 to the parabola is very close to the distance $\overline{OE}$ measured along y-axis, which indicates that the amplitude of forced vibrations for $\omega = p$ is very close to the maximum amplitude. For ω larger than p the amplitude of forced vibrations decreases indefinitely as the phase angle α increases and approaches the value π.*

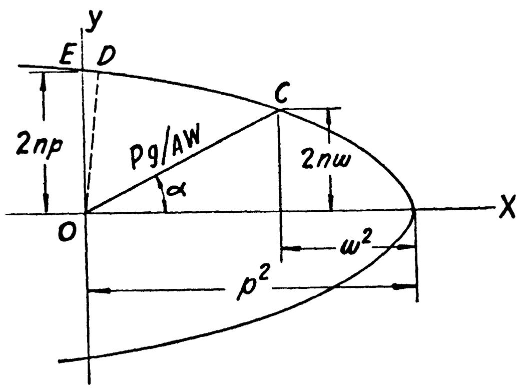

Fig. 30.

We have discussed thus far only the second part of the general expression (c) for motion of the body in Fig. 1, which represents the *steady* forced vibrations and which will be established only after the interval of time required to damp out the free vibration, produced at the beginning of the action of the disturbing force. If we are interested in motion which the body performs at the beginning of the action of the disturbing force the general expression for motion,

$$x = e^{-nt}(C_1 \cos p_1 t + C_2 \sin p_1 t) + A \sin (\omega t - \alpha), \tag{p}$$

must be used and the constants of integration C_1 and C_2 must be determined from the initial conditions. Assume, for instance, that for $t = 0$, $x = 0$ and $\dot{x} = 0$, i.e., the body is at rest at the instant when the disturbing force $P \sin \omega t$ begins to act. Then by using expression (p) and its derivative with respect to time we obtain

$$C_1 = A \sin \alpha, \qquad C_2 = \frac{nA \sin \alpha - \omega A \cos \alpha}{p_1}$$

by substituting which in eq. (p) the general expression for the motion of

* This graphical representation of forced vibrations is due to C. Runge, see paper by v. Sanden Ingenieur Archiv, vol. I, p. 645, 1930.

the body is obtained. For the case of a small damping and far from resonance the phase-angle α is small and we can take $C_1 = 0$, $C_2 = - \omega A/p_1$. The motion (p) is represented then by the following approximate expression

$$x = -\frac{\omega A e^{-nt}}{p_1} \sin p_1 t + A \sin \omega t. \qquad (q)$$

Thus on steady forced vibrations of amplitude A and with a circular frequency ω free vibrations, sometimes called *transient*, with a frequency p_1 and with a gradually damped out amplitude are superposed.

If the frequencies ω and p_1 are close together the phenomena of beating, discussed in article 3, will appear, but due to damping this beating will gradually die out and only steady forced vibrations will remain.

PROBLEMS

1. Determine the amplitude of forced vibrations produced by an oscillator, fixed at the middle of a beam, Fig. 19, at a speed 600 r.p.m. if $P = 1$ lb. the weight concentrated at the middle of the beam is $W = 1000$ lb. and produces statical deflection of the beam equal to $\delta_{st} = .01$ in. Neglect the weight of the beam and assume that damping is equivalent to a force acting at the middle of the beam, proportional to the velocity and equal to 100 lb. at a velocity of 1 in. per sec. Determine also the amplitude of forced vibration at resonance ($\omega = p$).

Solution. $\omega^2 = 400\pi^2$; $c = 100$

$$p^2 = 38600,$$

$$n = \frac{cg}{2W} = \frac{100 \times 386}{2 \times 1000} = 19.3,$$

$$P\omega^2 = 1 \cdot \omega^2 = 400\pi^2 \text{ lbs.},$$

$$A = \frac{Pg\omega^2}{W} \cdot \frac{1}{\sqrt{(p^2 - \omega^2)^2 + 4n^2\omega^2}} =$$

$$= \frac{400\pi^2 \times 386}{1000\sqrt{(38600 - 400\pi^2)^2 + 4 \times 19.3^2 \times 400\pi^2}} = .0439 \text{ in.},$$

if $\omega = p$,

$$A = \frac{Pgp^2}{W}\frac{1}{2np} = \frac{38600 \times 386}{1000 \times 2 \times 19.3 \times \sqrt{38600}} = 1.965 \text{ in.}$$

2. For the previous problem plot the curves representing the amplitude of forced vibration and the maximum velocity of the vibrating body W as functions of the ratio ω/p.

3. Investigate the effect of damping on the readings of the instrument shown in Fig. 14.

Assuming that the vibratory motion of the point of suspension A is given by $x_1 = a \sin \omega t$, the equation of motion of the suspended weight, by using notations (25), is

$$\ddot{x} + 2n\dot{x} + p^2 x = \frac{akg}{W} \sin \omega t.$$

Substituting ak for P in expression (33), the forced vibration becomes

$$x = \frac{a}{\sqrt{\left(1 - \frac{\omega^2}{p^2}\right)^2 + \frac{4n^2\omega^2}{p^4}}} \sin (\omega t - \alpha) = a\beta \sin (\omega t - \alpha), \tag{r}$$

where β is the magnification factor.

The instrument measures the difference of the displacements x_1 and x and we obtain

$$x_1 - x = a \sin \omega t - \beta a \sin (\omega t - \alpha).$$

The two terms on the right side of this equation can be added together by using rotating vectors $\overline{OC}$ of magnitude a and $\overline{OD}$ of magnitude βa as shown in Fig. 31. The geo-

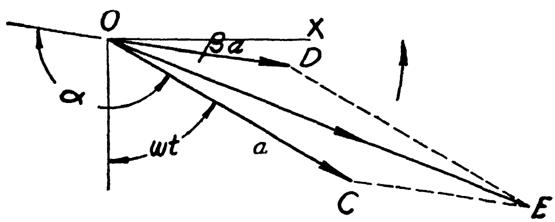

FIG. 31.

metrical sum OE of these two vectors gives us the amplitude of the relative motion $x_1 - x$. From the triangle OCE this amplitude is

$$A = a\sqrt{\beta^2 - 2\beta \cos \alpha + 1}. \tag{s}$$

It depends not only on the magnification factor β but also on the phase angle α.

In the case of instruments used for measuring amplitudes of vibrations (see Art. 4) the frequency ω is large in comparison with p, β is small, α approaches the value π and the amplitude, given by expression (s), is approximately equal to $a(1 + \beta)$. Substituting for β its value from eq. (r) and neglecting damping we find

$$A = a\left(1 + \frac{1}{\frac{\omega^2}{p^2} - 1}\right) = \frac{a}{1 - \frac{p^2}{\omega^2}},$$

which is approximately equal to a.

In the case of instruments used for measuring accelerations ω is small in comparison with p, α is small also and expression (s) approaches the value $a(\beta - 1)$. Substituting again for β its value and neglecting damping,* we get in this case

$$A = a\left(\frac{1}{1-\frac{\omega^2}{p^2}} - 1\right) = \frac{a}{\frac{p^2}{\omega^2} - 1},$$

which is approximately equal to $a\omega^2/p^2$ and proportional to the maximum acceleration.

4. Solve the problem 1 in Art. 4, see p. 22, assuming that there is damping in the spring. The damping force is proportional to vertical velocity of the body W and is equal to 1 lb. per unit mass of the body at the velocity 1 in. per sec. Calculate the amplitude of forced vibration at resonance ($p = \omega$). At what position of the wheel on the wave is the body in its highest position.

10. Spring Mounting of Machines with Damping Considered.—In our previous discussion of spring mounting of machines, Art. 5, it was assumed that there is no damping and the supporting springs are perfectly elastic. Such conditions are approximately realized in the case of helical steel springs, but if leaf springs or rubber and cork padding are used damping is considerable and cannot any longer be neglected. In the case of such imperfect springs it can be assumed that the spring force consists of two parts, one, proportional to the spring elongation, is an elastic force and the other, proportional to the velocity is a damping force. This condition can be realized, for instance, by taking a combination of perfect springs and a dash pot as shown in Fig. 32. Considering the case discussed in article 5 and calculating what portion of the disturbing force is transmitted to the foundation we have now to take into account not only the elastic force but also the force of damping. From Fig. 28 we see that these two forces act with a phase difference of 90 degrees and that the maximum of their resultant is

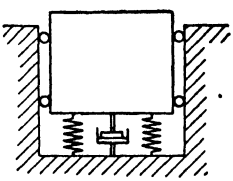

FIG. 32

$$A\sqrt{k^2 + c^2\omega^2} = Ak\sqrt{1 + \frac{4n^2\omega^2}{p^4}}, \tag{a}$$

where A is the amplitude of forced vibration, k is the spring constant and $c = 2nW/g$ is the damping force when the velocity is equal to unity. Substituting for A its value from formula (33) and taking, as in Art. 5, the

* Since the impressed motion is often not a simple sine motion and may contain higher harmonics with frequencies in the vicinity of the resonance of the instrument it is usual practice to have in accelerometers a considerable viscous damping, say taking $.5 < n/p < 1$.

disturbing force $P\omega^2 \sin \omega t$, we find that the maximum force transmitted to the foundation is

$$\frac{P\omega^2 \sqrt{1 + \dfrac{4n^2\omega^2}{p^4}}}{\sqrt{\left(1 - \dfrac{\omega^2}{p^2}\right)^2 + \dfrac{4n^2\omega^2}{p^4}}}. \tag{b}$$

Assuming that ω is large in comparison with p and at the same time the ratio n/p is small we find that the result (b) differs from what was found in Art. 5 principally by the presence of the term $4n^2\omega^2/p^4$ under the radical of the numerator.

Taking, as in problem 1, p. 26, $\omega = 60\pi$, $p^2 = 225$, $P = 1$ lb. and assuming $2n = 1$, we find

$$\sqrt{1 + 4n^2\omega^2/p^4} = 1.305, \quad \sqrt{\left(1 - \frac{\omega^2}{p^2}\right)^2 + \frac{4n^2\omega^2}{p^4}} = 156.9,$$

and the force transmitted to the foundation is

$$\frac{(60\omega)^2\, 1.305}{156.9} = 296 \text{ lb.}$$

which is about 30 per cent larger than we obtained before by neglecting damping.

The ratio of the force transmitted to the foundation (b) to the disturbing force $P\omega^2$ determines the *transmissibility*. It is equal to

$$\sqrt{1 + 4n^2\omega^2/p^4} : \sqrt{(1 - \omega^2/p^2)^2 + 4n^2\omega^2/p^4}, \tag{c}$$

and its magnitude depends not only on the ratio ω/p but also on the ratio n/p.

As a second example let us consider a single-phase electric generator. In this case the electric forces acting between the rotor and stator produce on the stator a pulsating torque which is represented by the equation

$$M_t = M_0 + M_1 \sin \omega t, \tag{d}$$

where ω is the double angular velocity of the rotor and M_0 and M_1 are constants.

If the stator is rigidly attached to the foundation the variable reactions due to pulsating torque may produce very undesirable vibrations. To reduce these reactions the stator is supported by springs as shown in Fig. 33.* The constant portion M_0 of the torque is directly transmitted to the

* See C. R. Soderberg, Electric Journal, vol. 21, p. 160, 1924.

foundation and produces constant reactions which can be readily obtained from the equations of statics. We have to consider only the variable portion $M_1 \sin \omega t$. Under the action of this variable moment the stator is subjected to rotatory vibrations with respect to the torque axis. If φ denotes the angle of rotation during these vibrations and k the spring constant which in this case represents the torque which, if applied statically, produces an angle of rotation of the stator equal to one radian, the moment of the reactions acting on the stator during vibration will be $-k\varphi$ and the equation of motion is

$$I\ddot{\varphi} + c\dot{\varphi} + k\varphi = M_1 \sin \omega t, \tag{e}$$

in which I is moment of inertia of the stator with respect to the torque axis and c is the magnitude of the damping couple for an angular velocity equal to unity. Using notations

$$\frac{c}{I} = 2n, \qquad \frac{k}{I} = p^2, \tag{f}$$

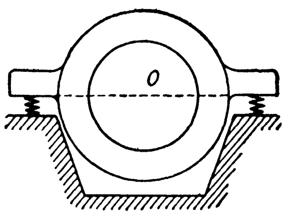

FIG. 33.

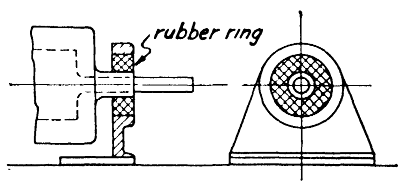

FIG. 34.

we bring equation (e) to the form of equation 32 and we can use the general expression (33) for the amplitude of forced vibration, it being only necessary to substitute in this expression M_1 instead of P. Multiplying this amplitude with the spring constant k we obtain the maximum value of the variable torque due to deformation of the springs. To this torque we must add the variable torque due to damping. Using the same reasoning as in the previous problem we finally obtain the maximum variable torque transmitted to foundation from expression (b) by substituting in it M_1 instead of $P\omega^2$.

The use of elastic supports in the case of single phase electric motors and generators has proved very successful. In the case of large machines the springs usually consist of steel beams. In small motors such as used in domestic appliances the required elasticity of supports is obtained by placing rubber rings between the rigid supports and the rotor bearings which are in this case rigidly built into the stator as shown in Fig. 34. The rubber ring firmly resists any lateral movement of the bearing since any radial compression of the rubber ring requires a circumferential expansion which is prevented by friction forces between the ring and the rigid support. At the same time any rotation of the stator produces in the rubber ring only shearing deformations which do not require a change in volume and the rubber in such case is very flexible and has on the transmission of the pulsating torque the same effect as the springs shown in Fig. 33.

We have another example of the use of elastic supports in the case of automobile internal combustion engines. Here again we deal with a pulsating torque which in the case of a rigidly mounted engine will be transmitted to the car. By introducing an elastic mounting, such that the engine may have low frequency rotary vibrations about the torque axis, a considerable improvement can be obtained.

11. Free Vibrations with Coulomb Damping.—As an example of vibrations with constant damping let us consider the case shown in Fig. 35. A

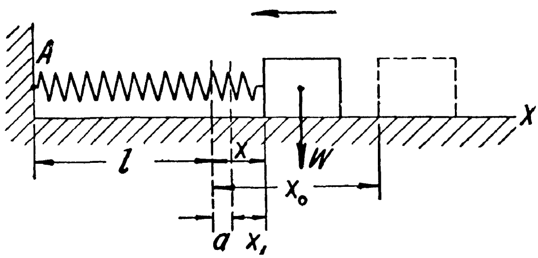

Fig. 35.

body W attached by a spring to a fixed point A slides along the horizontal dry surface with a vibratory motion. To write the equation of motion let us assume that the body is brought to its extreme right position and released. Then under the action of the tensile force in the spring it begins to move towards the left as shown. The forces which it is necessary to consider are: (1) the force in the spring, and (2) the friction force. Denoting by x the displacement of the body from the position at which the spring is unstretched and taking the positive direction of the x-axis, as shown in the figure, the spring force is $-kx$. The friction force in the case of a dry surface is constant. It acts in the direction opposite to the motion, i.e., in this case, in the positive direction of the x-axis. Denoting this force by F, the equation of motion becomes

$$\frac{W}{g}\ddot{x} = -kx + F, \tag{a}$$

or, by introducing notations

$$\frac{kg}{W} = p^2, \quad \frac{F}{k} = a, \tag{b}$$

we obtain

$$\ddot{x} + p^2(x - a) = 0, \tag{c}$$

where a has a simple physical meaning, namely, it represents the statical elongation of the spring which would be produced by the friction force F. Equation (c) can be brought in complete agreement with the eq. (3) (p. 2) for free vibrations without damping by introducing a new variable

$$x_1 = x - a, \tag{d}$$

which means that the distances will now be measured not from the position when the spring is unstretched but from the position when it has an elongation equal to a. Then, substituting x from (d) into eq. (c) we obtain

$$\ddot{x}_1 + p^2 x_1 = 0. \tag{e}$$

The solution of this equation, satisfying the initial conditions, is

$$x_1 = (x_0 - a) \cos pt, \tag{f}$$

where x_0 denotes the initial displacement of the body from the unstressed position. This solution is applicable as long as the body is moving to the left as assumed in the derivation of equation (a). The extreme left position will be reached after an interval of time equal to π/p, when $x_1 = -(x_0 - a)$ and the distance of the body from the unstressed position is $x_0 - 2a$. From this discussion it is seen that the time required for half a cycle of vibration is the same as in the case of free vibration without damping, thus the frequency of vibration is not effected by a constant damping. At the same time, considering the two extreme positions of the body defined by distances x_0 and $x_0 - 2a$, it can be concluded that during half a cycle the amplitude of vibrations is diminished by $2a$.

Considering now the motion of the body from the extreme left position to the right, and applying the same reasoning, it can be shown that during the second half of the cycle a further diminishing of the amplitude by the quantity $2a$ will occur. Thus the decrease of the amplitude follows the law of arithmetical progression. Finally, the load W will remain in one of its extreme positions as soon as the amplitude becomes less than a, since at such a position the friction force will be sufficient to balance the tensile force of the spring.

This vibratory motion again can be visualized by using rotating vectors. To obtain the motion corresponding to the first half of a cycle, eq. (*f*), we use vector O_1B_1, Fig. 36, of magnitude $x_0 - a$ rotating with a constant angular velocity p about the center O_1, which is displaced to the right with respect to the unstressed position O by the amount a. For the second half of the cycle we use the vector O_2B_2 of magnitude $x_0 - 3a$ and rotating with constant speed p around the center O_2, which is displaced from O to the left by the amount a, and so on. In this way we get a kind of a spiral, and the intersection point of this spiral with the x-axis in the interval O_1O_2 gives the final position of the body.

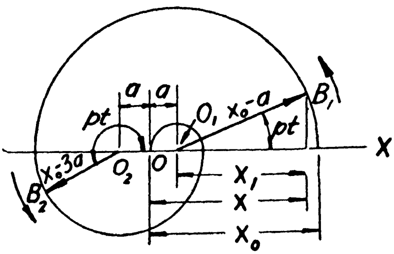

FIG. 36

PROBLEMS

1. The body in Fig. 35 is displaced from the unstressed position by the amount $x_0 = 10$ in., with the tensile force in the spring at this displacement, equal to $5W = 10$ lb., and then released without initial velocity. How long will the body vibrate and at what distance from the unstressed position will it stop if the coefficient of friction is ¼.

Solution. The friction force in this case is $F = W/4 = .5$ lb., spring constant $k = 1$ lb. per in., $a = \frac{1}{2}$ in. Hence the amplitude diminishes by 1 in. per each half a cycle and the body will stop after 5 cycles at the unstressed position. The period of one oscillation is $\tau = 2\pi\sqrt{\delta_{st}/g} = 2\pi\sqrt{2/386}$ and the total time of oscillation is $10\pi\sqrt{2/386} = 2.26$ sec.

2. What must be the relation between the spring constant k, the friction force F and the initial displacement x_0 to have the body stop at the unstressed position.

Answer. $\dfrac{x_0k}{F}$ must be an even number.

3. Determine the coefficient of friction for the case shown in Fig. 35 if a tensile force equal to W produces an elongation of the spring equal to ¼ in. and the initial amplitude $x_0 = 25$ in. is reduced to .90 of its value after 10 complete cycles.

Solution. The amplitude of vibration due to friction is reduced after each cycle by $4a = \dfrac{4F}{k}$ and since after 10 cycles it is reduced by 2.5 in. we have

$$10\frac{4F}{k} = \frac{10F}{W} = 2.5 \text{ in.}$$

Hence $F = \frac{1}{4}W$ and the coefficient of friction is equal to ¼.

4. For determining the coefficient of dry friction the device shown in Fig. 37 is used.*

* Such a device has been used about 30 years ago in the Friction Laboratory of the Polytechnical Institute in S. Petersburg.

A prismatical bar rests on two equal discs rotating with equal speeds in opposite directions. If the bar is displaced from the position of equilibrium and released, it begins to perform harmonic oscillations by moving back and forth along its axis. Prove that the coefficient of Coulomb friction between the materials of the bar and of the discs is given by formula

$$\mu = \frac{4\pi^2 a}{g\tau^2}$$

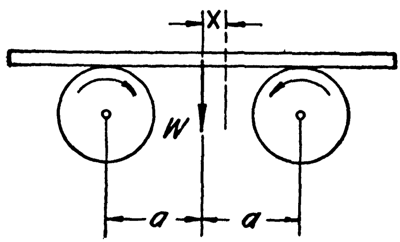

FIG. 37.

in which a is half the distance between the centers of the discs and τ is the period of oscillations of the bar.

Solution. If the bar is displaced from the middle position by the amount x the pressures on the discs are $W(a + x)/2a$ and $W(a - x)/2a$, the corresponding difference in friction forces is $F_1 - F_2 = \dfrac{\mu W}{a} x$ and is directed toward the axis of symmetry. It is the same as the force in a spring with elongation x and having a spring constant equal to $\mu W/a$. Hence the period of oscillation, from eq. 5 is

$$\tau = 2\pi\sqrt{\frac{W}{kg}} = 2\pi\sqrt{\frac{a}{\mu g}}$$

from which the formula given above for the coefficient of friction follows.

12. Forced Vibrations with Coulomb's Damping and Other Kinds of Damping.

—From the discussion of the previous article it is seen that to take care of the change in direction of the constant friction force F it is necessary to consider each half cycle separately. This fact complicates a rigorous treatment of the problem of forced vibration, but an approximate solution can be obtained without much difficulty.* In practical applications we are principally interested in the magnitude of steady

* This approximate method has been developed by L. S. Jacobsen, Trans. Am. Soc. Mech. Engrs., Vol. 52, p. 162, 1930. See also A. L. Kimball, Trans. Am. Soc. Mech. Engrs., Vol. 51, p. 227, 1930. The rigorous solution of the problem has been given by J. P. Den Hartog, Trans. Am. Soc. Mech. Engrs., Vol. 53, p. 107, 1931. See also Phil. Mag., Vol. 9, p. 801, 1930.

forced vibrations and this magnitude can be obtained with sufficient accuracy by assuming that the forced vibration in the case of a constant damping force F is a simple harmonic motion, as in the case of viscous damping, and by replacing the constant damping force by an *equivalent* viscous damping, such that the amount of energy dissipated per cycle will be the same for both kinds of damping.

Let $P \sin \omega t$ be the disturbing force and assume that the steady forced vibration is given by the equation

$$x = A \sin (\omega t - \alpha). \tag{a}$$

Between two consecutive extreme positions the vibrating body travels a distance $2A$, thus the work done per cycle against the constant friction force, representing the dissipated energy, is

$$4AF. \tag{b}$$

If instead of constant friction we have a viscous damping the corresponding value of the dissipated energy is given by formula (38), p. 45, and the magnitude of the equivalent viscous damping is determined from the equation

$$\pi c A^2 \omega = 4AF \tag{c}$$

from which

$$c = \frac{4F}{\pi A \omega}. \tag{d}$$

Thus the magnitude of the equivalent viscous damping depends not only on F but also on the amplitude A and the frequency ω of the vibration. Using notations (25), p. 32 and substituting in expression (33)

$$\frac{2n}{p^2} = \frac{c}{k} = \frac{4F}{\pi A k \omega},$$

we obtain for the amplitude of the forced vibration with equivalent viscous damping the following expression

$$\frac{P}{k} \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{p^2}\right)^2 + \left(\frac{4F}{\pi A k}\right)^2}}.$$

This expression represents the amplitude A in eq. (a), hence the equation for determining A is

$$\frac{P}{k}\frac{1}{\sqrt{\left(1-\frac{\omega^2}{p^2}\right)^2+\left(\frac{4F}{\pi Ak}\right)^2}}=A,$$

from which

$$A=\pm\frac{P}{k}\cdot\frac{\sqrt{1-(4F/\pi P)^2}}{1-\omega^2/p^2}. \tag{40}$$

The first factor on the right side represents *static deflection* and the second is the *magnification factor*. We see that this factor has a real value only if

$$F/P<\pi/4. \tag{e}$$

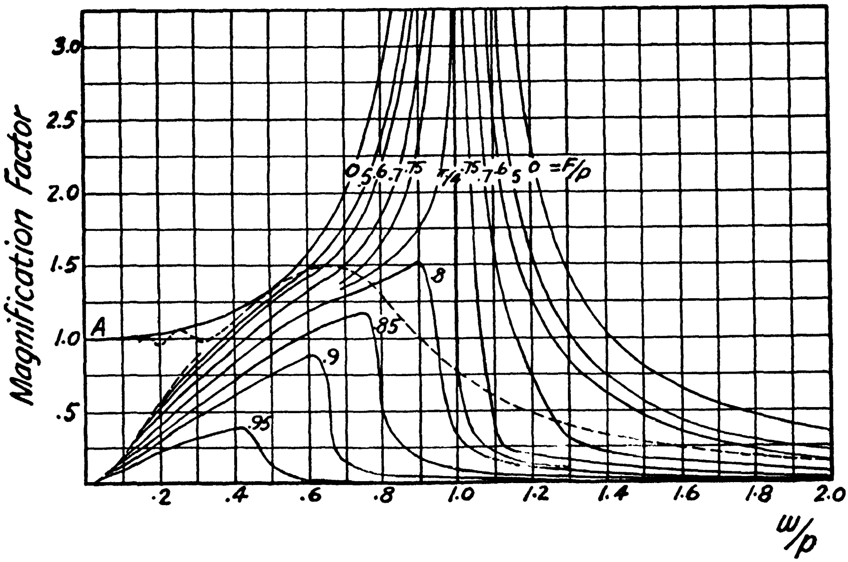

Fig. 38

In practical applications, where we are usually dealing with small frictional force, this condition is satisfied and we find that the magnification factor depends on the value of the ratio ω/p. Values of this factor, for various values of the ratio F/P, are plotted against ω/p in Fig. 38 * It

* This figure and the two following are taken from the above mentioned Den Hartog's exact solution By the dotted line the limit is indicated above which a non-stop oscillatory motion occurs. Below that limit the motion is more complicated and the curves shown in the figure can be obtained only by using the exact solution

is seen that in all cases in which condition (e) is satisfied the magnification factor becomes infinity at resonance ($p = \omega$), which means that in this case even with considerable friction the amplitude at resonance tends to infinity. This fact can be explained if we consider the dissipation of energy and the work produced by the disturbing force. In the case of viscous damping the energy dissipated per cycle, eq. (38), increases as the square of the amplitude. At the same time the work produced per cycle by the disturbing force (eq. 37) increases in proportion to the amplitude. Thus the finite amplitude is obtained by intersection of the parabola with a straight line as shown in Fig. 39. In the case of constant friction the dissipated energy is proportional to A, eq. (b), and in Fig. 39 it will be represented by a straight line the slope of which is smaller than the slope of the line OE, if condition (e) is satisfied, hence there will always be an excess of input and the amplitude increases indefinitely.

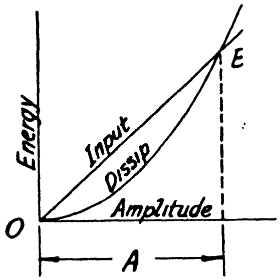

FIG 39.

By substituting the value of the equivalent damping (eq. d) into eq. (34) and using eq. (40) we obtain the equation

$$\tan \alpha = \pm \frac{4F}{\pi P} \frac{1}{\sqrt{1 - \left(\frac{4F}{\pi P}\right)^2}} \qquad (f)$$

from which the phase angle α can be calculated. The angle does not vary with the ratio ω/p and only at resonance ($\omega = p$) it changes its value abruptly The exact solution shows that the phase angle varies somewhat with the ratio ω/p as shown in Fig 40

The described approximate method of investigating forced vibrations can be used also in general, when the friction force is any function of the velocity In each particular case it is only necessary to calculate the corresponding equivalent damping by using an equation similar to eq. (c). Assuming for example that the friction force is represented by a function $f(\dot{x})$, this equation becomes

$$\pi c A^2 \omega = \int_0^\tau f(\dot{x}) \dot{x}\, dt \qquad (g)$$

Substituting for x its expression from eq (a) the value of c can be always calculated.

Take, as an example, a combination of Coulomb's friction and viscous friction. Then

$$f(\dot{x}) = \pm F + c_1\dot{x}.$$

Substituting in eq. (g) we find

$$\pi c A^2\omega = 4AF + \pi c_1 A^2 \omega$$

from which

$$c = \frac{4F}{\pi A\omega} + c_1.$$

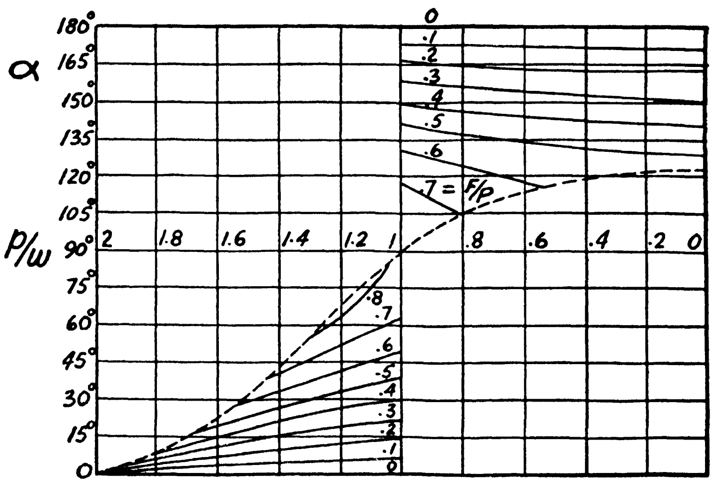

FIG. 40.

Proceeding with this value of c as before we obtain for determining the amplitude A the equation

$$A^2\left[\left(1 - \frac{\omega^2}{p^2}\right)^2 + \frac{c_1{}^2\omega^2}{k^2}\right] + 2A\,\frac{4Fc_1\omega}{\pi k^2} + \left(\frac{4F}{\pi k}\right)^2 - \frac{P^2}{k^2} = 0. \qquad (h)$$

When $c_1 = 0$ this equation gives for A expression (40). When $F = 0$ we get for A expression (33). For any given values of F and c_1, the amplitude of forced vibrations can be readily obtained from equation (h).

PROBLEMS

1. For the case considered in problem 1 of the previous article find the amplitude of forced steady vibration if the frequency of the disturbing force $P \sin \omega t$ is $1\frac{1}{2}$ per sec. and its maximum value is equal to W. Answer. 3.50 inches.

2. Develop an approximate equation for the amplitude of steady forced vibration if the damping force is proportional to the square of velocity.*

Solution. Assuming that the damping force is given by the expression $c_1(\dot{x})^2$ and taking one quarter of a cycle, starting, from the middle position, the dissipated energy is .

$$c_1\omega^3A^3\int_0^{\pi/2\omega}\cos^3\omega t\,dt = \tfrac{2}{3}c_1\omega^2A^3$$

and eq. (g) becomes

$$\pi cA^2\omega = \tfrac{8}{3}c_1\omega^2A^3$$

from which

$$c = \frac{8}{3\pi}c_1\omega A$$

and equation for calculating A becomes

$$A^4 + A^2\frac{P^2}{k^2}\cdot\frac{\left(1-\dfrac{\omega^2}{p^2}\right)^2}{\dfrac{\omega^4}{p^4}\cdot c_1^2\left(\dfrac{8}{3\pi}\right)^2\dfrac{p^4}{k^4}P^2} - \frac{\dfrac{P^4}{k^4}}{\dfrac{\omega^4}{p^4}\cdot c_1^2\left(\dfrac{8}{3\pi}\right)^2\dfrac{p^4}{k^4}P^2} = 0$$

or

$$A^4 + A^2k^2\frac{\left(1-\dfrac{\omega^2}{p^2}\right)^2}{c_1^2\omega^4\left(\dfrac{8}{3\pi}\right)^2} - \frac{P^2}{\omega^4c_1^2\left(\dfrac{8}{3\pi}\right)^2} = 0.$$

13. Balancing of Rotating Machines.—One of the most important applications of the theory of vibrations is in the solution of balancing problems. It is known that a rotating body does not exert any variable disturbing action on the supports when the axis of rotation coincides with one of the principal axes of inertia of the body. It is difficult to satisfy this condition exactly in the process of manufacturing because due to errors in geometrical dimensions and non-homogeneity of the material some irregularities in the mass distribution are always present. As a result of this variable disturbing forces occur which produce vibrations. In order to remove these vibrations in machines and establish quiet running conditions, balancing becomes necessary. The importance of balancing becomes especially great in the case of high speed machines. In such cases the slightest unbalance may produce a very large disturbing force. For instance, at 1800 r.p.m. an unbalance equal to one pound at a radius of 30 inches produces a disturbing force equal to 2760 lbs.

* Free vibrations with damping proportional to the square of velocity was studied by W. E. Milne, University of Oregon Publications, No. 1 (1923) and No. 2 (1929). The tables attached to these papers will be useful in studying such vibrations. For forced vibrations we have the approximate solution given by L. S. Jacobsen developed in the previously mentioned paper.

In order to explain the various conditions of unbalance a rotor shown in Fig. 41, *a* will now be considered.* Imagine the rotating body divided into two parts by any cross section *mn*. The three following typical cases of unbalance may arise:

1. The centers of gravity of both parts may be in the same axial plane and on the same side of the axis of rotation as shown in Fig. 41, *b*. The center of gravity C of the whole body will consequently be in the same plane at a certain distance from the axis of rotation. This is called "*static unbalance*," because it can be detected by a statical test. A statical balancing test consists of putting the rotor with the two ends of its shaft on absolutely horizontal, parallel rails. If the center of gravity of the whole

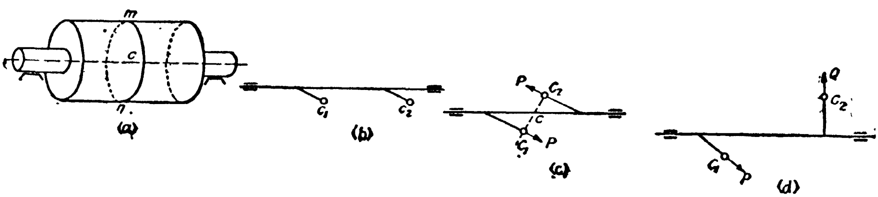

FIG. 41.

rotor is in the axis (Fig. 41, *c*) the rotor will be in static equilibrium in any position; if the center is slightly off the shaft, as in Fig. 40, *b*, it will roll on the rails till the center of gravity reaches its lowest position.

2. The centers of gravity of both parts may be in the same axial plane but on opposite sides of the axis of rotation as shown in Fig. 41, *c*, and at such radial distances that the center of gravity C of the whole body will be exactly on the axis of rotation. In this case the body will be in balance under static conditions, but during rotation a disturbing couple of centrifugal forces P will act on the rotor. This couple rotates with the body and produces vibrations in the foundation. Such a case is called "*dynamic unbalance.*"

3. In the most general case the centers of gravity, C_1 and C_2, may lie in different axial sections and during rotation a system of two forces formed by the centrifugal forces P and Q will act on the body (see Fig. 41, *d*). This system of forces can always be reduced to a couple acting in an axial section and a radial force, i.e., static and dynamic unbalance will occur together.

It can be shown that in all cases complete balancing can be obtained by attaching to the rotor a weight in each of two cross sectional planes

* The rotor is considered as an absolutely rigid body and vibrations due to elastic deflections of it are neglected.

arbitrarily chosen. Consider, for instance, the case shown in Fig. 42. Due to unbalance two centrifugal forces P and Q act on the rotor during motion. Assume now that the weights necessary for balance must be located in the cross sectional planes I and II. The centrifugal force P can be balanced by two forces P_1 and P_2, lying with P in the same axial section. The magnitude of these forces will be determined from the following equations of statics

$$P_1 + P_2 = P,$$
$$P_1 a = P_2 b.$$

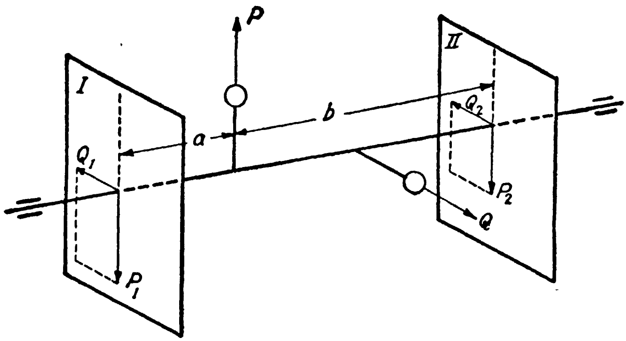

FIG. 42.

In the same manner the force Q can be balanced by the forces Q_1 and Q_2. The resultant of P_1 and Q_1 in plane I, and the resultant of P_2 and Q_2 in plane II will then determine the magnitudes and the positions of the correction weights necessary for complete balancing of the rotor. It is seen from this discussion that balancing can be made without any difficulty if the position and magnitude of the unbalance is known. For determining this unbalance various types of balancing machines are used and the fundamentals of these machines will now be discussed.

14. Machines for Balancing.—A balancing machine represents usually an arrangement in which the effects of any unbalance in the rotor which is under test may be magnified by resonance. There are three principal types of balancing machines: first, machines where the rotor rests on two independent pedestals such as the machines of Lawaczeck-Heymann, or the Westinghouse machine; second, machines in which the rotor rests on a vibrating table with an immovable fulcrum; third, balancing machines with a movable fulcrum.

The *machine* of *Lawaczeck-Heymann* consists mainly of two independent pedestals. The two bearings supporting the rotor are attached to springs, which allow vibrations of the ends of the rotor in a horizontal axial plane. One of the bearings is locked with the balancing being performed on the other end (see Fig. 43). Any unbalance will produce vibration of the rotor about the locked bearing as a fulcrum. In order to magnify these vibrations all records are taken at *resonance condition.* By a special motor the rotor is brought to a speed above the *critical* and then the motor power is shut off. Due to friction the speed of the rotor gradually decreases and as it passes through its *critical value* pronounced forced vibrations of the unlocked bearing of the rotor will be produced by any unbalance. The process of balancing then consists of removing these vibra-

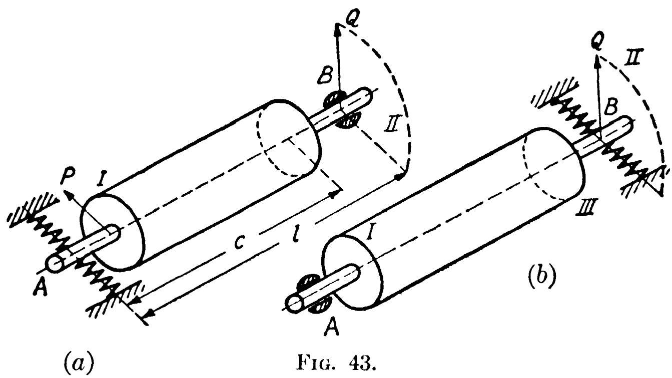

FIG. 43.

tions by attaching suitable correction weights. The most suitable planes for placing these weights are the ends of the rotor body, where usually special holes for such weights are provided along the circumference. By such an arrangement the largest distance between the correction weights is obtained; therefore the magnitude of these weights is brought to a minimum. When the plane for such correction weight has been chosen there still remain two questions to be answered, (1) the location of the correction weight and (2) its magnitude. Both these questions can be solved by trial. In order to determine the location, some arbitrary correction weight should be put in the plane of balancing and several runs should be made with the weight in different positions along the circumference of the rotor. A curve representing the variation in amplitude of vibrations, with the angle of location of the weight, can be so obtained. The minimum amplitude will then indicate the true location for the correction weight. In the same manner, by gradual changing the magnitude of the weight, the true magnitude of the correction weight can be established.

In order to simplify the process of determining the location of the correction weight, marking the shaft or recording the vibrations of the shaft end may be very useful. For marking the shaft a special indicator, shown in Fig. 44, is used in the Lawaczeck machine. During vibration the shaft presses against a pencil *ab* suitably arranged and displaces it into a position corresponding to the maximum deflection of the shaft end so that the end of the marking line on the surface of the shaft determines the angular position at the moment of maximum deflection. Assuming that at resonance the lag of the forced vibrations is equal to $\pi/2$, the location of the disturbing force will be 90° from the point where the marking ceases in the direction of rotation of the shaft. Now the true location for the correction weight can easily be obtained. Due to the fact that near the resonance condition the lag changes very sharply with the speed and also depends on the damping (see Fig. 27, p. 42) two tests are usually necessary for an accurate determination of the location of unbalance. By running the shaft alternately in opposite directions and marking the shaft as explained above the bisector between the two marks determines the axial plane in which the correction weights must be placed.

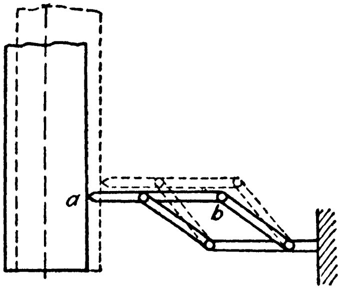

FIG. 44.

For obtaining the location of unbalance, by recording the vibrations of the face of the shaft end of the rotor, a special vibration recorder is used in the Lawaczeck machine. The recording paper is attached to the face of thè shaft and revolves with the rotor. The pencil of the indicator pressing against the paper performs displacements which are the magnified lateral displacements of the shaft end with respect to the immovable pedestal of the machine. In this manner a kind of a polar diagram of lateral vibrations of the shaft will be obtained on the rotating paper attached to the shaft end. By running the shaft twice, in two opposite directions, two diagrams on the rotating paper will be obtained. The axis of symmetry for these two diagrams determines the plane in which the correction weight must be placed.*

* A more detailed description of methods of balancing by using the Lawaczeck-Heymann machine can be found in the paper by Ernst Lehre: "Der heutige Stand der Auswuchttechnik," Maschinenbau, Vol. 16 (1922–1923), p. 62. See also the paper by E. v. Brauchisch, "Zur Theorie und experimentellen Prüfung des Auswuchtens," Zeitschr. f. Angw. Math. und Mech., Vol. 3 (1923), p. 61, and the paper by J. G. Baker and F. C. Rushing, The Journal of the Franklin Institute, Vol. 222, p. 183, 1936.

The procedure for balancing a rotor AB (see Fig. 43) will now be described. Assume first that the bearing B is locked and the end A of the rotor is free to vibrate in a horizontal axial plane. It has been shown (see p. 63) that in the most general case the unbalance can be represented by two centrifugal forces acting in two arbitrarily chosen planes perpendicular to the axis of the shaft. Let the force P in the plane I (see Fig. 43, a) and the force Q in the plane II through the center of the locked bearing B represent the unbalance in the rotor. In the case under consideration the force P only will produce vibrations. Proceeding as described above the force P can be determined and the vibrations can be annihilated by a suitable choice of correction weights. In order to balance the force Q, the bearing A must be locked and the bearing B made free to vibrate (see Fig. 43, b). Taking the plane III, for placing the correction weight and proceeding as before, the magnitude and the location of this weight can be determined. Let G denote the centrifugal force corresponding to this weight. Then from the equation of statics,

$$G \cdot c = Q \cdot l$$

and

$$Q = \frac{Gc}{l}. \tag{a}$$

It is easy to see that by putting the correction weight in the plane III, we annihilate vibrations produced by Q only under the condition that the bearing A is locked. Otherwise there will be vibrations due to the fact that the force Q and the force G are acting in two different planes II and III. In order to obtain complete balance one correction weight must be placed in each of the two planes I and III, such that the corresponding centrifugal forces G_1 and G_2 will have as their resultant the force $-Q$ equal and opposite to the force Q (Fig. 45). Then, from statics, we have,

$$G_1 - G_2 = Q,$$
$$G_2 \cdot b = Q \cdot a,$$

from which, by using eq. (a)

$$G_2 = \frac{Qa}{b} = \frac{Gac}{bl}, \tag{b}$$

$$G_1 = Q + G_2 = \frac{Gcd}{bl}. \tag{c}$$

It is seen from this that by balancing at the end B and determining in this manner the quantity G, the true correction weight for the plane III and the additional correction weight for the plane I can be found from equations (b) and (c) and complete balancing of the rotor will be obtained.

The large *Westinghouse machine** having a capacity of rotors weighing up to 300,000 pounds consists essentially of two pedestals mounted on a rigid bedplate, together with a driving motor and special magnetic clutch for rotating the rotor. A cross section of the pedestal consists of a solid part bolted to the rails of the bedplate and a pendulum part held in place by strong springs. The vertical load of the rotor is carried by a flexible

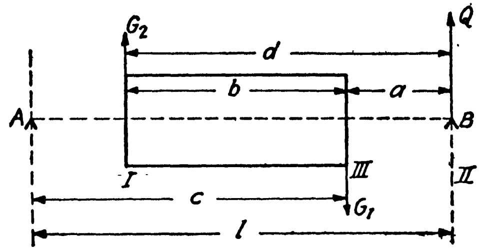

Fig. 45.

thin vertical plate, making a frictionless hinge. The rotor is brought to a speed above the critical speed of the bearing which can be controlled by changing the springs according to the weight of the rotor, and the magnetic clutch is disengaged. The rotor drifts slowly through this critical speed when observations of the oscillations produced by magnified effect of the unbalance are made.

The balancing is done by locking first one bearing and balancing the opposite end, and then locking the second end and balancing the corresponding opposite end. The balancing is done by a cut-and-try method, the time of balancing proper of large rotors being small when compared with the time of setting up and preparations for balancing. The additional correction weights are put into the balancing rings, the same as described with the Lawaczeck-Heymann machine.

Akimoff's Balancing Machine,† consists of a rigid table on which the rotor and the compensating device are mounted. The table is secured to the pedestals in such a way that it is free to vibrate, either about an

* L. C. Fletcher, "Balancing Large Rotating Apparatus," Electrical Journal, Vol. XXI, p. 5.

† Trans. A. S. M. E., Vol. 38 (1916).

axis parallel to the axis of the rotor or about an axis perpendicular to the axis of the rotor. In the first case static unbalance alone produces vibrations; in the second, both static and dynamic unbalance will cause vibration. Beginning with checking for static unbalance, the table must be supported in such a way as to obtain vibration about the axis parallel to the axis of rotation of the rotor. The method for determining the location and magnitude of unbalance consists in creating an artificial unbalance in some moving part of the machine to counteract the unbalance of the body to be tested. When this artificial unbalance becomes the exact counterpart of the unbalance in the body being tested, the whole unit ceases to vibrate and the magnitude and the angular plane of unbalance are indicated on the machine.

After removing the static unbalance of the rotor testing for dynamic unbalance can be made by re-arranging the supports of the table in such a manner as to have the axis of vibration perpendicular to the axis of rotation. The magnitude and the angular plane of dynamic unbalance will then be easily found in the same manner as explained above by introducing an artificial couple of unbalance in the moving part of the machine. It is important to note that all the static unbalance must be removed before checking for dynamic unbalance.

The Soderberg-Trumpler machine is an example of the third type.

When mass production balancing of small units is performed, the time per unit necessary for balancing is of great importance. The additional correction weights necessary with the previously described types (see p. 67) cause a loss of time. In order to eliminate these corrections, the fulcrum of the balancing table is movable in this machine. The body to be balanced is mounted in bearing blocks on a vibrating table supported by two spring members and a movable fulcrum. By placing the fulcrum axis in the plane of one of the balancing rings, say BB, the action of the theoretical unbalance weight in this plane is eliminated as far as its effect upon the motion of the vibrating table is concerned. This will now be produced by the unbalance in the other plane only. Then the force at AA is balanced, after which the fulcrum is moved to the position in the plane AA; then BB is balanced. It is evident that this balancing is final and does not require any correction. These machines are used mostly when small rotors are balanced.

On this principle, an automatic machine is built by the Westinghouse Company for their small motor works.* In order to eliminate harmful

* W. E. Trumpler, "The Dynamic Balance of Small High Speed Armatures," Electric Journal, Vol. 22, 1925, p. 34.

damping in friction joints, knife edges were replaced by flexible spring members. The table oscillates horizontally, being carried on a vertical stem presenting a torsionally flexible axis. The table proper is moved in guides in such a way that one weight correction plane can be brought for balancing in line with the axis of the vertical stem.

For "automatic" balancing, the table is supplied with an unbalance compensating head coupled to the rotor. The counter-balancing is done by two electrically operated small clutches. The movable weights in the head produce a counter balancing couple. One clutch shifts the weights apart, increasing the magnitude of this couple; another clutch changes the angular position of the counter-balancing couple with respect to the rotor. Two switches mounted in front of the machine actuate the clutches. It is easy in a very short time, a fraction of a minute, to adjust the counter-balancing weights in a way that the vibration of the table is brought to zero. Indicators on the balancing head show then the amount and location of unbalance, and the necessary correction weights are inserted into the armature.*

Balancing in the Field.—Experience with large high speed units shows that while balancing carried out on the balancing machine in the shop may show good results, nevertheless this testing is usually done at comparatively low speed and in service where the operating speeds are high, unbalance may still be apparent due to slight changes in mass distribution.† It is therefore necessary also to check the balancing condition for normal operating speed. This test is carried out, either in the shop where the rotor is placed for this purpose on rigid bearings or in the field after it is assembled in the machine. The procedure of balancing in such conditions can be about the same as described above in considering the Lawaczeck balancing machine. This consists in consecutive balancing of both ends of the rotor. In correcting the unbalance at one end, it is assumed that vibrations of the corresponding pedestal are produced only by the unbalance at this end.‡ The magnitude and the location of the correction weight can then be found from measurements of the amplitudes of vibrations of the pedestal, which are recorded

* Recently several new types of balancing machines have been developed which reduce considerably the time required for balancing. It should be mentioned here the Leblanc-Thearle balancing machine described in Trans. Am. Soc. Mech. Engrs., Vol. 54, p. 131, 1932; the Automatic Balancing Machine of Spaeth-Losenhausen and the method of balancing rotors by means of electrical networks recently developed by J. G. Baker and F. C. Rushing, Journal of the Franklin Institute, Vol. 222, p. 183, 1936.

† See Art. 50.

‡ This assumption is accurate enough in cases of rotors of considerable length

by a suitable instrument. Four measurements are necessary for four different conditions of the rotor in order to have sufficient data for a complete solution of the problem. The first measurement must be made for the rotor in its initial condition and the three others for the rotor with some arbitrary weights placed consecutively in three different holes of the balancing ring of the rotor end at which the balancing is being performed. A rough approximation of the location of the correction weight can be found by marking the shaft of the rotor in its initial condition as explained before (p. 66). The three trial holes must be taken near the location found in this manner (Fig. 46, *a*).

On the basis of these four measurements the determination of the unbalance can now be made on the assumption that the amplitudes of vibration of the pedestal are proportional to the unbalance. Let $A0$ (Fig. 46, *b*) be the vector representing the unknown original unbalance and let 01, 02, 03 be the vectors corresponding to the trial corrections I, II and III put into the balancing ring of the rotor end during the second, third and fourth runs, respectively. Then vectors $A1$, $A2$, $A3$ (Fig. 46, *b*) represent the resultant unbalances for these three runs. These vectors, according to the assumption made are proportional to the amplitudes of vibration of the pedestal measured during the respective runs.

When balancing a rotor, the magnitudes and directions of 01, 02, 03 (Fig. 46, *b*) are known and a network as shown in Fig. 46, *c* by dotted lines can be constructed. Taking now three lengths $A'1'$, $A'2'$, and $A'3'$ proportional to the amplitudes observed during the trial runs and using the network, a diagram geometrically similar to that given in Fig. 46, *b*, can be constructed (Fig. 46, *c*). The direction $0A'$ gives then the location of the true correction and the length $0A'$ represents the weight of it to the same scale as $01'$, $02'$ and $03'$ represent the trial weights I, II and III, respectively. It should be noted that the length $0A'$ if measured to the same scale as the amplitudes $A'1'$, $A'2'$, $A'3'$, must give the amplitude of the initial vibration of the pedestal, this being a check of the solution obtained. In the photograph 46, *d* a simple device for the solution of this problem, consisting of four straps connected together by a hinge 0_1, is shown.* Taking now three lengths a_1, a_2 and a_3 on the straps proportional to the amplitudes observed during the trial runs and moving the ends of these straps along the radii of the correction weights such as radii $01'$, $02'$, $03'$ in Fig. 46, *c* it will make no difficulty to find a position of the system where all these three ends will be situated on the same broken line

* This device has been developed by G. B. Karelitz and proved very useful in field balancing. See "Power," Feb. 7 and 14, 1928.

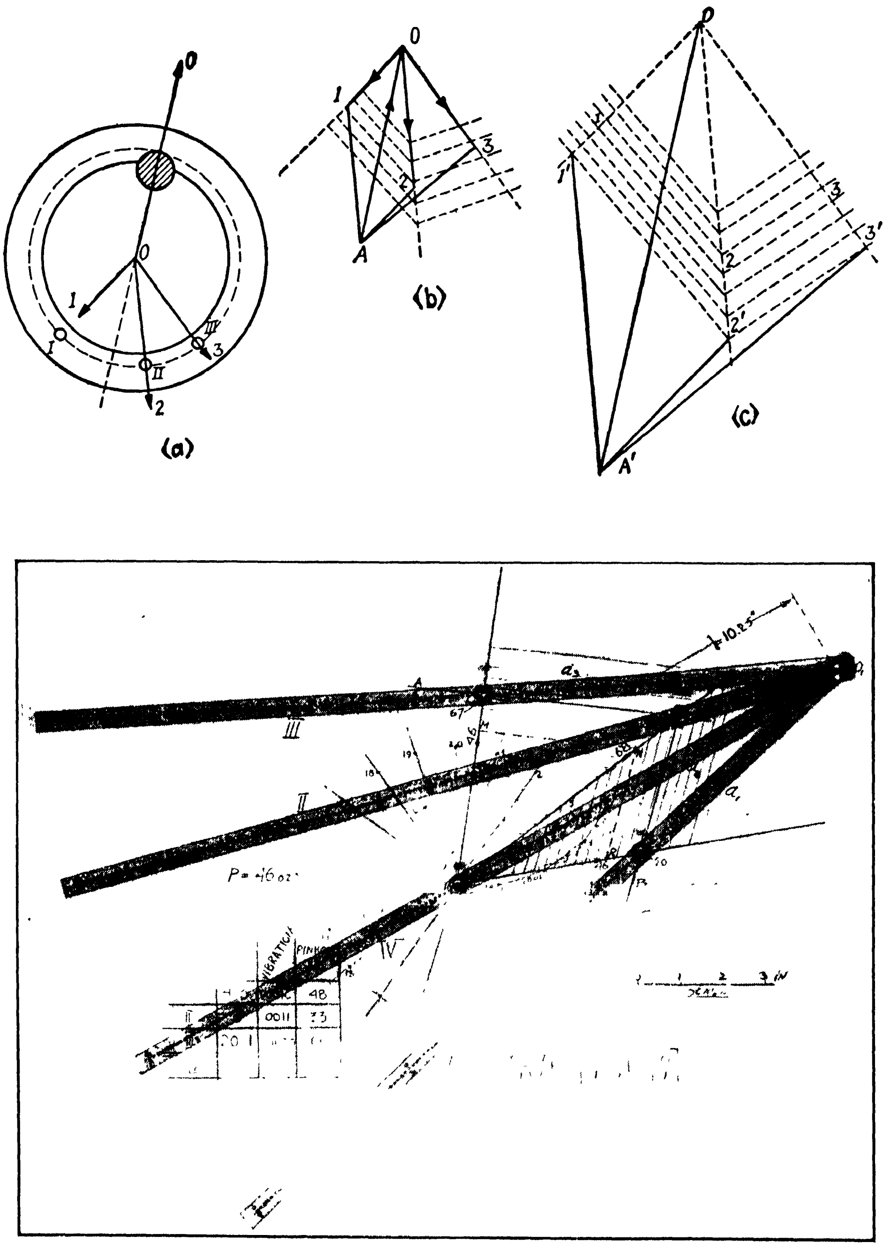

FIG 46.

of the net work such as line 1′2′3′. The corresponding vector 00_1 will then determine the position and the magnitude of the true correction weight.

The second method of balancing is based on measurement of amplitude and angle of lag of vibration produced by unbalance. It is assumed that an angle of lag of vibration behind the disturbing unbalance force is constant, while the unbalance is changed when the correction weights are placed into the balancing ring of the high-speed rotor. This angle of lag may be measured in a rough way by simply marking or scribing the shaft.

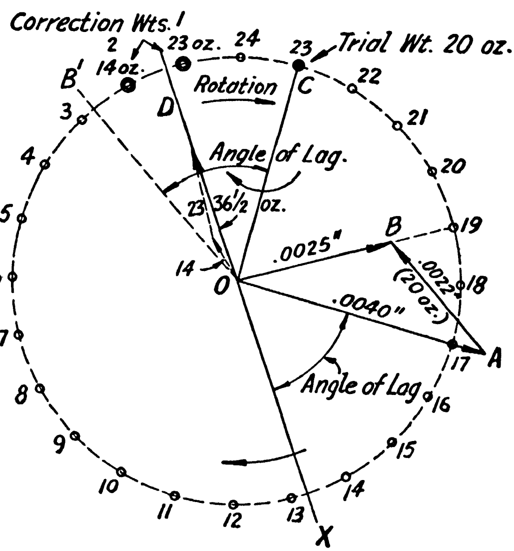

FIG. 47.

A rough estimate of the unbalance can be obtained by the Single-Direct Method.* The amplitude of vibration of a rotor bearing is observed first without any correction weights in the balancing ring, and the shaft is scribed. (This is done by painting the shaft with chalk and touching it as lightly as possible with a sharp tool while rotating at full speed.) After the rotor is stopped, the location of the "high spot" of this scribe mark is noted with respect to the balancing holes of the rotor. A correction weight is then placed in the balancing ring (preferably about 60° to 90° behind

* This method, suggested by B. Anoshenko, is described in the paper by T. C. Rathbone, Turbine Vibration and Balancing, Trans. A.S.M.E. 1929, paper APM-51-23.

this high spot). With the rotor brought up again to full speed, the amplitude is recorded and the shaft scribed again, the location of the new high spot being later noted. The method of determination of the unbalance will be demonstrated in an example. Assume twenty-four balancing holes in the rotor with no correction weights (Fig. 47). The amplitude is .004″ and the high spot is found to be in line with hole No. 17. After placing 20 ounces in hole No. 23, the amplitude changes to .0025″, and the high spot is found to be in line with hole No. 19. The diagram of Fig. 47 shows the construction for the location and determination of the correction weight. Vector OA to hole No. 17 represents the amplitude of .004″ to a certain scale. Vector OB to hole No. 19 to the same scale represents the amplitude .0025″. Vector AB shows then the variation in the vibration to the same scale. This variation was produced by the weight C placed in hole No. 23. Making OB' parallel to AB the angle COB' is then the angle of lag. The original disturbing unbalance force is evidently located at an angle $AOX = COB'$ ahead of the original high spot. The correction weight has to be placed in the direction OD opposite to OX. The magnitude of the necessary correction weight is 20 ounces times the ratio of OA to AB, or $36\frac{1}{2}$ ounces.

The scribing of the shaft is a very crude and unreliable operation and the method should be considered as satisfactory only for an approximate commercial determination of the unbalance. A more accurate result can be obtained by using a *phasometer* in measuring the angle of lag.*

15. Application of Equation of Energy in Vibration Problems.—In investigating vibrations the equation of energy can sometimes be used advantageously. Consider the system shown in Fig. 1. Neglecting the mass of the spring and considering only the mass of the suspended body, the kinetic energy of the system during vibration is

$$\frac{W}{2g}\dot{x}^2. \qquad (a)$$

The potential energy of the system in this case consists of two parts: (1) the potential energy of deformation of the spring and (2) the potential energy of the load W by virtue of its position. Considering the energy of deformation, the tensile force in the spring corresponding to any displacement x from the position of equilibrium, is $k(\delta_{st} + x)$ and the corresponding strain energy is $k(\delta_{st} + x)^2/2$. At the position of equilibrium

* This method is developed by T. C. Rathbone, see paper mentioned above, p. 73.

($x = 0$) this energy is $k\delta^2_{st}/2$. Hence the energy stored in the spring during the displacement x is

$$\frac{k(\delta_{st} + x)^2}{2} - \frac{k\delta^2_{st}}{2} = Wx + \frac{kx^2}{2}. \tag{b}$$

The energy due to position of the load diminishes during the displacement x by the amount Wx. Hence the total change of the potential energy of the system is

$$\frac{kx^2}{2}. \tag{c}$$

Due to the fact that the load W is always in balance with the initial tension in the spring produced by the static elongation δ_{st}, the total change in potential energy is the same as in the case shown in Fig. 35, in which, if we neglect friction, the static deflection of the spring is zero.

Having expressions (a) and (c) and neglecting damping the equation of energy becomes

$$\frac{W}{2g}\dot{x}^2 + \frac{kx^2}{2} = \text{const.} \tag{d}$$

The magnitude of the constant on the right side of this equation is determined by the initial conditions. Assuming that at the initial instant, $t = 0$, the displacement of the body is x_0 and the initial velocity is zero, the initial total energy of the system is $kx_0^2/2$ and equation (d) becomes

$$\frac{W}{2g}\dot{x}^2 + \frac{kx^2}{2} = \frac{kx_0^2}{2}. \tag{e}$$

It is seen that during vibration the sum of the kinetic and potential energy remains always equal to the initial strain energy. When in the oscillatory motion x becomes equal to x_0 the velocity $\dot{x}$ becomes equal to zero and the energy of the system consists of the potential energy only. When x becomes equal to zero, i.e., the vibrating load is passing through its middle position, the velocity has its maximum value and we obtain, from eq. (e),

$$\frac{W}{2g}\dot{x}^2_{\max} = \frac{kx_0^2}{2}. \tag{f}$$

Thus the maximum kinetic energy is equal to the strain energy stored in the system during its displacement to the extreme position, $x = x_0$.

In all cases in which it can be assumed that the motion of a vibrating body is a simple harmonic motion, which is usually correct for small vibrations,* we can use equation (f) for the calculation of the frequency of vibration. We assume that the motion is given by the equation $x = x_0 \sin pt$. Then $(\dot{x})_{\max} = x_0 p$. Substituting in eq. (f) we obtain

$$p^2 = \frac{kg}{W}. \tag{g}$$

This coincides with eq. (2), previously obtained, p. 2.

The use of eq. (f) in calculating frequencies is especially advantageous if instead of a simple problem, as in Fig. 1, we have a more complicated system. As an example let us consider the frequency of free vibrations of the weight W of an *amplitude meter* shown in Fig. 48.

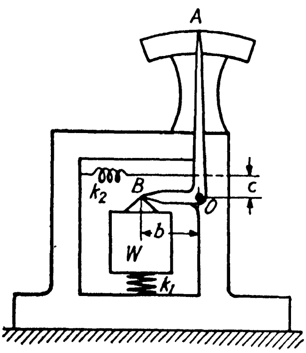

FIG. 48.

The weight is supported by a soft spring k_1 so that its natural frequency of vibration is low in comparison with the frequency of vibrations which are measured by the instrument. When the amplitude meter is attached to a body performing high frequency vertical vibrations the weight W, as explained before, see art. 4, remains practically immovable in space and the pointer A connected with W, indicates on the scale the magnified amplitude of the vibration. In order to obtain the frequency of the free vibrations of the instrument with greater accuracy, not only the weight W and the spring k_1, but also the arm AOB and the spring k_2 must be taken into consideration. Let x denote a small vertical displacement of the weight W from the position of equilibrium. Then the potential energy of the two springs with the spring constants k_1 and k_2 will be

$$\frac{k_1 x^2}{2} + \frac{k_2}{2}\left(\frac{c}{b}\right)^2 x^2. \tag{h}$$

The kinetic energy of the weight W will be, as before,

$$\frac{W}{2g}\dot{x}^2. \tag{k}$$

* Some exceptional cases are discussed in Chapter II.

The angular velocity of the arm AOB rotating about the point O is

$$\frac{\dot{x}}{b}$$

and the kinetic energy of the same arm is

$$\frac{I}{2}\frac{\dot{x}^2}{b^2}. \tag{l}$$

Now the equation of motion, corresponding to eq. (d) above, will be from (h), (k) and (l),

$$\left(\frac{W}{2g} + \frac{I}{2b^2}\right)\dot{x}^2 + \left(\frac{k_1}{2} + \frac{k_2}{2}\frac{c^2}{b^2}\right)x^2 = \text{const.}$$

We see that this equation has the same form as the equation (d); only instead of the mass W/g we have now the *reduced* mass

$$\frac{W}{g} + \frac{I}{b^2}$$

and instead of the spring constant k we have the *reduced spring constant* $k_1 + k_2(c^2/b^2)$.

As another example let us consider torsional vibrations of a shaft one end of which is fixed and to the other end is attached a disc connected

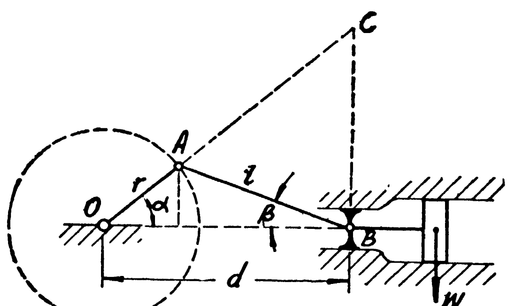

Fig. 49.

with a piston as shown in Fig. 49. We consider only small rotatory oscillations about its middle position given by the angle α. If φ is the angle of twist of the shaft at any instant, the potential energy of the system, which in this case is the strain energy of torsion of the shaft, is equal to $k\varphi^2/2$, where k is the spring constant of the shaft. In calculating the kinetic energy of the system we have to consider the kinetic energy of the rotating parts, equal to $I\dot{\varphi}^2/2$ and also the kinetic energy of the reciprocating masses.* In calculating the kinetic energy of the reciprocating masses, the total weight of which we call by W, it is necessary to have the expression for the velocity of these masses during torsional vibration. The angular velocity $\dot{\theta}$ of the connecting rod AB with respect to the instantaneous center C, Fig. 49, can be obtained from the consideration of the velocity of the point A. Considering this point as belonging to the disc its velocity during vibration is $r\dot{\varphi}$. The velocity of the same point, as belonging to the connecting rod, is $\overline{AC}\ \dot{\theta}$ and we obtain

$$r\dot{\varphi} = \overline{AC}\ \dot{\theta} = \frac{l \cos \beta}{\cos \alpha}\dot{\theta}$$

where l is the length of the connecting rod and β its angle of inclination to the horizontal. From this equation

$$\dot{\theta} = \frac{r\dot{\varphi} \cos \alpha}{l \cos \beta}$$

and the velocity of the reciprocating masses is

$$\dot{x} = \dot{\theta}\ \overline{BC} = \dot{\theta}\ (l \cos \beta + r \cos \alpha) \tan \alpha = r\dot{\varphi} \sin \alpha \left(1 + \frac{r \cos \alpha}{l \cos \beta}\right). \qquad (m)$$

We obtain also from the figure

$$r \sin \alpha = l \sin \beta.$$

Hence

$$\sin \beta = \frac{r}{l} \sin \alpha; \quad \cos \beta = \sqrt{1 - \frac{r^2}{l^2} \sin^2 \alpha} \approx 1 - \frac{1}{2}\frac{r^2}{l^2} \sin^2 \alpha. \qquad (n)$$

If the ratio r/l is small we can assume with sufficient accuracy that $\cos \beta \approx 1$. Then the velocity of the reciprocating masses is

$$\dot{x} \approx r\, \dot{\varphi} \sin \alpha \left(1 + \frac{r}{l} \cos \alpha\right) \qquad (o)$$

* The mass of the connecting rod can be replaced by two masses, $m_1 = I_1/l^2$ at the crankpin and $m_2 = m - m_1$ at the crosshead, where m is the total mass of the connecting rod and I_1 its moment of inertia about the center of the crosshead. This is the usual way of replacing the connecting rod, see Max Tolle, "Regelung der Kraftmaschinen," 3d Ed., p. 116, 1921.

and the total kinetic energy of the system is

$$\frac{I\dot{\varphi}^2}{2} + \frac{W}{2g} r^2\dot{\varphi}^2 \sin^2 \alpha \left(1 + \frac{r}{l} \cos \alpha\right)^2.$$

The energy equation in this case becomes

$$\frac{I\dot{\varphi}^2}{2} + \frac{W}{2g} r^2\dot{\varphi}^2 \sin^2 \alpha \left(1 + \frac{r}{l} \cos \alpha\right)^2 = \frac{k\varphi_0^2}{2}.$$

The effect of the reciprocating masses on the frequency of torsional vibrations is the same as the increase in the moment of inertia of the disc obtained by adding to the circumference of the disc of a reduced mass equal to

$$\frac{W}{g} \sin^2 \alpha \left(1 + \frac{r}{l} \cos \alpha\right)^2 \cdot \tag{p}$$

It is seen that the frequency depends on the magnitude of the angle α. When α is zero or π, the reciprocating masses do not effect the frequency and the effect becomes maximum when α is approximately equal to $\pi/2$.

PROBLEMS

1. Calculate the frequencies of small vibrations of the pendulums shown in Fig. 50*a*, *b*, *c*, by using the equation of energy. Neglect the mass of the bars and assume that in each case the mass of the weight W is concentrated in its center.

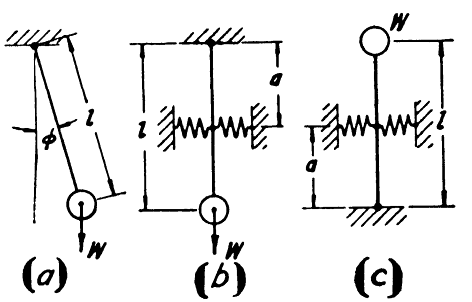

FIG. 50.

Solution. If φ is the angle of inclination of the pendulum, Fig. 50*a*, and l its length, the kinetic energy of the pendulum is $W\dot{\varphi}^2l^2/2g$. The change in potential energy of the pendulum is due to vertical displacement $l(1 - \cos \varphi) \approx l\varphi^2/2$ of the weight W and equation of energy becomes

$$\frac{W\dot{\varphi}^2l^2}{2g} + \frac{Wl\varphi^2}{2} = \text{const.} \tag{r}$$

Assuming motion $\varphi = \varphi_0 \sin pt$ and writing an equation, similar to eq. (*f*) we obtain the circular frequency

$$p = \sqrt{\frac{g}{l}}.$$

In the case shown in Fig. 50*b* the strain energy of the springs must be added to the potential energy of the weight W in writing the equation of energy. If k is the spring constant, by taking into consideration both springs, the strain energy of springs is $k(a\varphi)^2/2$ and, instead of eq. (*r*), we obtain

$$\frac{W\dot{\varphi}^2 l^2}{2g} + (Wl + ka^2)\frac{\varphi^2}{2} = \text{const.} \tag{s}$$

and the frequency of vibrations becomes

$$p = \sqrt{\frac{g}{l}\left(1 + \frac{ka^2}{Wl}\right)}.$$

In the case shown in Fig. 50*c*, the potential energy of the weight W, at any lateral displacement of the pendulum from vertical position, decreases and by using the same reasoning as before we obtain

$$p = \sqrt{\frac{g}{l}\left(\frac{ka^2}{Wl} - 1\right)}.$$

It is seen that we obtain a real value for p only if

$$\frac{ka^2}{Wl} > 1 \quad \text{and} \quad W < \frac{ka^2}{l}.$$

If this condition is not satisfied the vertical position of equilibrium of the pendulum is not stable.

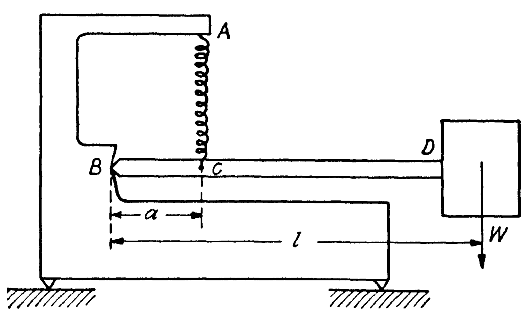

FIG. 51.

2. For recording of ship vibrations a device shown in Fig. 51 is used.* Determine the frequency of vertical vibrations of the weight W if the moment of inertia I of this weight, together with the bar BD about the fulcrum B is known.

Solution. Let φ be the angular displacement of the bar BD from its horizontal position of equilibrium and k the constant of the spring, then the energy stored during

* This is O. Schlick's pallograph, see Trans. Inst. Nav. Arch., Vol. 34, p. 167, 1893.

this displacement is $ka^2\varphi^2/2$ and the kinetic energy of the system is $I\dot{\varphi}^2/2$. The energy equation becomes

$$\frac{I\dot{\varphi}^2}{2} + \frac{ka^2\varphi^2}{2} = \text{const.}$$

Proceeding as in the case of eq. (d) we get for circular frequency the expression

$$p = \sqrt{\frac{ka^2}{I}}.$$

If we neglect the mass of the bar BD and assume the mass of the weight W concentrated in its center $I = Wl^2/g$ and the frequency becomes

$$p = \sqrt{\frac{ka^2g}{Wl^2}} = \sqrt{\frac{a}{l}\,\frac{g}{\delta_{st}}},$$

where $\delta_{st} = Wl/ak$ is statical elongation of the spring. Comparing this with eq. (2) it can be concluded that for the same elongation of the spring the horizontal pendulum has a much lower frequency than the device shown in Fig 1 provided that the ratio a/l is sufficiently small. A low frequency of the vibration recorder is required in this case since the frequency of natural vibration of a large ship may be comparatively low, and the frequency of the instrument must be several times smaller than the frequency of vibrations which we are studying (see art. 4).

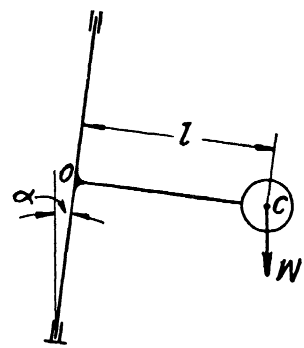

FIG. 52.

3. Figure 52 represents a heavy pendulum the axis of rotation of which makes a small angle α with the vertical. Determine the frequency of small vibration considering only the weight W which is assumed to be concentrated at its mass center C.

Solution. If φ denotes a small angle of rotation of the pendulum about the inclined axis measured from the position of equilibrium the corresponding elevation of the center C is

$$l(1 - \cos\varphi)\sin\alpha \approx \frac{l\varphi^2}{2}\alpha$$

and the equation of energy becomes

$$\frac{W}{2g}l^2\dot{\varphi}^2 + \frac{Wl\varphi^2\alpha}{2} = \text{const.}$$

and the circular frequency of the pendulum is

$$p = \sqrt{\frac{g\alpha}{l}}.$$

It is seen that by choosing a small angle α the frequency of the pendulum may be made very low. This kind of pendulum is used sometimes in recording earthquake vibrations. To get two components of horizontal vibrations two instruments such as shown in Fig. 52 are used, one for the N.–S. component and the other for the E.–W. component.

4. For recording vertical vibrations the instrument shown in Fig. 53 is used, in which a rigid frame AOB carrying the weight W can rotate about an axis through O perpendicular to the plane of the figure. Determine the frequency of small vertical vibrations of the weight if the moment of inertia I of the frame together with the weight

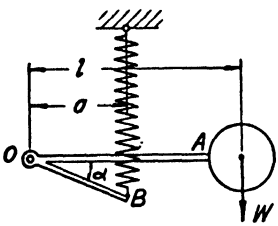

FIG. 53.

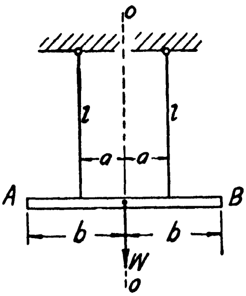

FIG. 54.

about the axis through O and the spring constant k are known and all the dimensions are given.

Answer. $p = \sqrt{\dfrac{ka^2}{I}}.$

5. A prismatical bar AB suspended on two equal vertical wires, Fig 54, performs small rotatory oscillations in the horizontal plane about the axis oo. Determine the frequency of these vibrations.

Solution. If φ is the angle of rotation of the bar from the position of equilibrium, the corresponding elevation of the bar is $a^2\varphi^2/2l$ and the energy equation becomes

$$\frac{I\dot{\varphi}^2}{2} + \frac{Wa^2\varphi^2}{2l} = \text{const.}$$

Taking $I = Wb^2/3g$ we obtain the frequency

$$p = \sqrt{\frac{3ga^2}{lb^2}}.$$

6. What frequency will be produced if the wires in the previous problem will be placed at an angle β to the axis oo.

Answer. $p = \sqrt{\cos\beta}\sqrt{\dfrac{3ga^2}{lb^2}}.$

7. The journals of a rotor are supported by rails curved to a radius R, Fig. 55. Determine the frequency of small oscillations which the rotor performs when rolling without sliding on the rails.

Solution. If φ is the angle defining the position of the journals during oscillations and r is the radius of the journals, the angular velocity of the rotor during vibrations is

$\dot{\varphi}(R - r)/r$, the velocity of its center of gravity is $(R - r)\dot{\varphi}$ and the vertical elevation of this center is $(R - r)\varphi^2/2$. Then equation of energy is

$$\frac{I\dot{\varphi}^2(R - r)^2}{2r^2} + \frac{W(R - r)^2\dot{\varphi}^2}{2g} + \frac{W(R - r)\varphi^2}{2} = \text{const.}$$

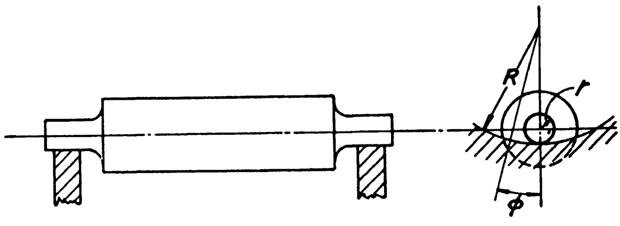

FIG. 55.

where I is the moment of inertia of the rotor with respect to its longitudinal axis. For the frequency of vibration we obtain

$$p^2 = \frac{Wr^2}{\left(I + W\frac{r^2}{g}\right)(R - r)}.$$

8. A semi-circular segment of a cylinder vibrates by rolling without sliding on a horizontal plane, Fig. 56. Determine the frequency of small vibrations.

Answer. Circular frequency is

$$p = \sqrt{\frac{cg}{i^2 + (r - c)^2}}$$

where r is the radius of the cylinder, $c = \overline{OC}$ is the distance of center of gravity and $i^2 = Ig/W$ the square of the radius of gyration about centroidal axis.

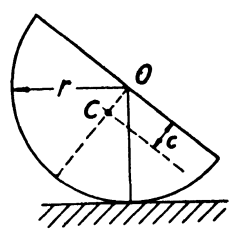

FIG. 56.

16. Rayleigh Method.—In all the previously considered cases, such as shown in Figs. 1, 4, and 7, by using certain simplifications the problem was reduced to the simplest case of vibration of a system with one degree of freedom. For instance, in the arrangement shown in Fig. 1, the mass of the spring was neglected in comparison with the mass of the weight W, while in the arrangement shown in Fig. 4 the mass of the beam was neglected and again in the case shown in Fig. 7 the moment of inertia of the shaft was neglected in comparison with the moment of inertia of the disc. Although these simplifications are accurate enough in many practical cases, there are technical problems in which a more detailed consideration of the accuracy of such approximations becomes necessary.

In order to determine the effect of such simplifications on the frequency of vibration an approximate method developed by Lord Rayleigh* will now be discussed. In applying this method some assumption regarding the configuration of the system during vibration has to be made. The frequency of vibration will then be found from a consideration of the energy of the system. As a simple example of the application of Rayleigh's method we take the case shown in Fig. 1 and discussed in Art. 15.

Assuming that the mass of the spring is small in comparison with the mass of the load W, the *type of vibration* will not be substantially affected by the mass of the spring and with a sufficient accuracy it can be assumed that the displacement of any cross section of the spring at a distance c from the fixed end is the same as in the case of a massless spring, i.e., equal to

$$\frac{xc}{l}, \tag{a}$$

where l is the length of the spring.

If the displacements, as assumed above, are not affected by the mass of the spring, the expression for the potential energy of the system will be the same as in the case of a massless spring, (see eq. (c), p. 75) and only the kinetic energy of the system has to be reconsidered. Let w denote the weight of the spring per unit length. Then the mass of an element of the spring of length dc will be wdc/g and the corresponding kinetic energy, by using eq. (a), becomes

$$\frac{w}{2g}\left(\frac{\dot{x}c}{l}\right)^2 dc.$$

The complete kinetic energy of the spring will be

$$\frac{w}{2g}\int_0^l \left(\frac{\dot{x}c}{l}\right)^2 dc = \frac{\dot{x}^2}{2g}\frac{wl}{3}.$$

This must be added to the kinetic energy of the weight W; so that the equation of energy becomes

$$\frac{\dot{x}^2}{2g}\left(W + \frac{wl}{3}\right) + \frac{kx^2}{2} = \frac{kx_0{}^2}{2}. \tag{b}$$

Comparing this with eq. (e) of the previous article it can be concluded that in order to estimate the effect of the mass of the spring on

* See Lord Rayleigh's book "Theory of Sound," 2d Ed., Vol. I, pp. 111 and 287.

the period of natural vibration it is only necessary to add one-third of the weight of the spring to the weight W.

This conclusion, obtained on the assumption that the weight of the spring is very small in comparison with that of the load, can be used with sufficient accuracy even in cases where the weight of the spring is of the same order as W. For instance, for $wl = .5W$, the error of the approximate solution is about $^1/_2\%$. For $wl = W$, the error is about $\frac{3}{4}\%$. For $wl = 2W$, the error is about 3%.*

As a second example consider the case of vibration of a beam of uniform cross section loaded at the middle (see Fig. 57). If the weight wl of the beam is small in comparison with the load W, it can be assumed with sufficient accuracy that the deflection curve of the beam during vibration has the same shape as the statical deflection curve. Then, denoting by x the displacement of the load W during vibration, the displacement of any element wdc of the beam, distant c from the support, will be,

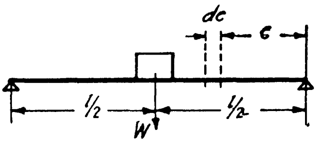

FIG. 57.

$$x \cdot \frac{3cl^2 - 4c^3}{l^3}.$$

The kinetic energy of the beam itself will be,

$$2\int_0^{l/2} \frac{w}{2g}\left(\dot{x}\,\frac{3cl^2 - 4c^3}{l^3}\right)^2 dc = \frac{17}{35}\,wl\,\frac{\dot{x}^2}{2g}. \tag{41}$$

This kinetic energy of the vibrating beam must be added to the energy $W\dot{x}^2/2g$ of the load concentrated at the middle in order to estimate the effect of the weight of the beam on the period of vibration, i.e., the period of vibration will be the same as for a massless beam loaded at the middle by the load

$$W + (17/35)wl.$$

It must be noted that eq. (41) obtained on the assumption that the weight of the beam is small in comparison with that of the load W, can be used in all practical cases. Even in the extreme case where $W = 0$ and where the assumption is made that $(17/35)wl$ is concentrated at the middle of the beam, the accuracy of the approximate method is sufficiently

* A more detailed consideration of this problem is given in Art. 52.

close. The deflection of the beam under the action of the load $(17/35)wl$ applied at the middle is,

$$\delta_{st} = \frac{17}{35}\, wl \cdot \frac{l^3}{48EI}\,.$$

Substituting this in eq. (5) (see p. 3) the period of the natural vibration is

$$\tau = 2\pi\sqrt{\frac{\delta_{st}}{g}} = .632\sqrt{\frac{wl^4}{EIg}}\,.$$

The exact solution for this case * is

$$\tau = \frac{2}{\pi}\sqrt{\frac{wl^4}{EIg}} = .637\sqrt{\frac{wl^4}{EIg}}\,.$$

It is seen that the error of the approximate solution for this limiting case is less than 1%.

The same method can be applied also in the case shown in Fig. 58. Assuming that during the vibration the shape of the deflection curve of the beam is the same as the one produced by a load statically applied at the end and denoting by x the vertical displacement of the load W the kinetic energy of the cantilever beam of uniform cross section will be,

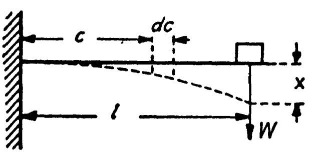

FIG. 58.

$$\int_0^l \frac{w}{2g}\left(\dot{x}\,\frac{3c^2l - c^3}{2l^3}\right)^2 dc = \frac{33}{140}\, wl\, \frac{\dot{x}^2}{2g}. \tag{42}$$

The period of vibration will be the same as for a massless cantilever beam loaded at the end by the weight,

$$W + (33/140)wl.$$

This result was obtained on the assumption that the weight wl of the beam is small in comparison with W, but it is also accurate enough for cases where wl is not small. Applying the result to the extreme case where $W = 0$ we obtain

$$\delta_{st} = \frac{33}{140}\, wl\, \frac{l^3}{3EI}\,.$$

* See Art. 56.

The corresponding period of vibration will be

$$\tau = 2\pi\sqrt{\frac{\delta_{st}}{g}} = \frac{2\pi}{3.567}\sqrt{\frac{wl^4}{EIg}}. \tag{c}$$

The exact solution for the same case is *

$$\tau = \frac{2\pi}{3.515}\sqrt{\frac{wl^4}{EIg}}. \tag{d}$$

It is seen that the error of the approximate solution is about 1½%. For the case $W = 0$ a better approximation can be obtained. It is only necessary to assume that during the vibration the shape of the deflection curve of the beam is the same as the one produced by a uniformly distributed load. The deflection y_0 at any cross section distant c from the built-in section will then be given by the following equation,

$$y_0 = x_0\left\{-1/3 + (4/3)\frac{c}{l} + 1/3\left(1 - \frac{c}{l}\right)^4\right\}, \tag{e}$$

in which

$$x_0 = \frac{wl^4}{8EI}$$

represents the deflection of the end of the cantilever.

The potential energy of bending will be

$$V = \frac{w}{2}\int_0^l y_0 dc = \frac{8}{5}\cdot\frac{EIx_0^2}{l^3}.$$

The kinetic energy of the vibrating beam is

$$T = \frac{1}{2}\int_0^l \frac{w}{g}\dot{y}^2 dc.$$

Taking (see p. 75)

$$y = y_0 \cos pt \quad \text{and} \quad (\dot{y})_{\text{max.}} = y_0 p.$$

The equation for determining p will be (see eq. (f), p. 75)

$$\frac{1}{2}\int_0^l \frac{w}{g}(y_0 p)^2 dc = \frac{8}{5}\frac{EIx_0^2}{l^3}.$$

* See Art. 57

Substituting (e) for y_0 and performing the integration, we obtain

$$p = 3.530\sqrt{\frac{EIg}{wl^4}}\,.$$

The corresponding period of vibration is

$$\tau = \frac{2\pi}{3.530}\sqrt{\frac{wl^4}{EIg}}. \qquad (f$$

Comparing this result with the exact solution (d) it can be conclude that in this case the error of the approximate solution is only about ½%

It must be noted that an elastic beam represents a system with a infinitely large number of degrees of freedom. It can, like a string perform vibrations of various types. The choosing of a definite shap for the deflection curve in using Rayleigh's method is equivalent to intro ducing some additional constraints which reduce the system to one havin one degree of freedom. Such additional constraints can only increas the rigidity of the system, i.e., increase the frequency of vibration. I all cases considered above the approximate values of the frequencie as obtained by Rayleigh's method are somewhat higher than their exac values.*

In the case of torsional vibrations (see Fig. 7) the same approximat method can be used in order to calculate the effect of the inertia of th shaft on the frequency of the torsional vibrations. Let i denote th moment of inertia of the shaft per unit length. Then assuming that th type of vibration is the same as in the case of a massless shaft the angl of rotation of a cross section at a distance c from the fixed end of th shaft is $c\varphi/l$ and the kinetic energy of one element of the shaft will be

$$\frac{idc}{2}\left(\frac{c\dot{\varphi}}{l}\right)^2.$$

The kinetic energy of the entire shaft will be

$$\frac{i}{2}\int_0^l\left(\frac{c\dot{\varphi}}{l}\right)^2 dc = \frac{\dot{\varphi}^2}{2}\frac{il}{3}. \qquad (43$$

This kinetic energy must be added to the kinetic energy of the disc i order to estimate the effect of the mass of the shaft on the frequency o vibration, i.e., the period of vibration will be the same as for a massles

* A complete discussion of Rayleigh's method can be found in the book by G Temple and W. G. Bickley, "Rayleigh's Principle," Oxford University Press, 1933.

shaft having at the end a disc, the moment of inertia of which is equal to

$$I + il/3.$$

The application of Rayleigh's method for calculating the critical speed of a rotating shaft will be shown in the following article.

PROBLEMS

1. Determine the frequency of natural vibrations of the load W supported by a beam AB, Fig. 59, of constant cross section (1) assuming that the weight of the beam can be neglected; (2) taking the weight of the beam into consideration and using Rayleigh's method.

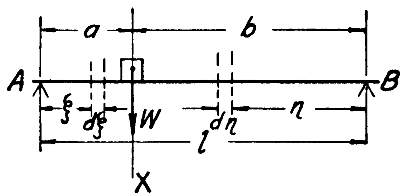

FIG. 59.

Solution. If a and b are the distances of the load from the ends of the beam the static deflection of the beam under the load is $\delta = Wa^2b^2/3lEI$. Taking for the spring constant the expression $k = 3lEI/a^2b^2$ and neglecting the mass of the beam the circular frequency of vibration is obtained from the equation of energy (see p. 75)

$$\frac{W}{2g}\dot{x}^2_{\max} = \frac{kx_0^2}{2} \tag{g}$$

in which $\dot{x}_{\max} = x_0p$. Hence

$$p = \sqrt{\frac{kg}{W}} = \sqrt{\frac{3lEIg}{Wa^2b^2}}. \tag{h}$$

To take the mass of the beam into account we consider the deflection curve of the beam under static action of the load W. The deflection at any point of the left portion of the beam at the distance ξ from the support A is

$$x_1 = \frac{W\xi b}{6lEI}[a(l + b) - \xi^2]. \tag{i}$$

For the deflection at any point to the right of the load W and at a distance η from the support B we have

$$x_2 = \frac{Wa\eta}{6lEI}[b(l + a) - \eta^2]. \tag{j}$$

Applying Rayleigh's method and assuming that during vibration the maximum velocity of any point of the left portion of the beam at a distance ξ from the support A is given by the equation

$$(\dot{x}_1)_{\max} = \dot{x}_{\max}\frac{x_1}{\delta} = \dot{x}_{\max}\frac{\xi}{2a^2b}[a(l + b) - \xi^2]$$

in which $\dot{x}_{\max}$ is the maximum velocity of the load W, we find that to take into account the mass of the left portion of the beam we must add to the left side of the equation (*g*) the quantity

$$\frac{w\dot{x}^2_{\max}}{2g}\int_0^a\left(\frac{x_1}{\delta}\right)^2 d\xi = \frac{w\dot{x}^2_{\max}}{2g}\int_0^a \frac{\xi^2}{4a^4b^2}[a(l+b)-\xi^2]^2\,d\xi$$
$$= \dot{x}^2_{\max}\frac{wa}{2g}\left[\frac{1}{3}\frac{l^2}{b^2}+\frac{23}{105}\frac{a^2}{b^2}-\frac{8}{15}\frac{al}{b^2}\right]. \qquad (k)$$

In the same manner considering the right portion of the beam we find that we must add to the left side of eq. (*g*) the expression

$$\frac{\dot{x}^2_{\max}wb}{2g}\left[\frac{1}{12}\frac{(l+a)^2}{a^2}+\frac{1}{28}\frac{b^2}{a^2}-\frac{1}{10}\frac{b(l+a)}{a^2}\right]. \qquad (l)$$

The equation of energy becomes

$$\frac{(W+\alpha wa+\beta wb)}{2g}\dot{x}^2_{\max} = \frac{kx_0^2}{2}$$

where α and β denote the quantities in the brackets of expressions (*k*) and (*l*) and we obtain for the frequency of vibration the following formula

$$p = \sqrt{\frac{3lEIg}{(W+\alpha aw+\beta bw)a^2b^2}}. \qquad (m)$$

2. Determine the frequency of the natural vertical vibrations of the load W supported by a frame hinged at A and B, Fig. 60*a*, assuming that the three bars of the frame have the same length and the same cross section and the load is applied at the middle of the bar CD. In the calculation (1) neglect the mass of the frame; (2) consider the mass of the frame by using Rayleigh's method.

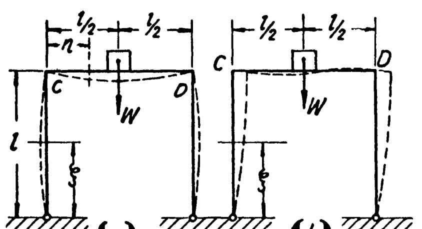

FIG. 60.

Solution. By using the known formulas for deflections of beams we find that the bending moments at the joints C and D are equal to $3Wl/40$. The deflections of vertical bars at a distance ξ from the bottom is

$$x_1 = \frac{3Wl^2\xi}{240EI}\left(1-\frac{\xi^2}{l^2}\right). \qquad (n)$$

The deflections of the horizontal bar to the left of the load is

$$x_2 = \frac{W\eta}{48EI}(3l^2-4\eta^2) - \frac{3}{80}\frac{Wl}{EI}\eta(l-\eta). \qquad (o)$$

The deflection under the load W is

$$\delta = (x_2)_{\eta=\frac{l}{2}} = \frac{11}{960}\frac{Wl^3}{EI}.$$

By neglecting the mass of the frame we find the frequency

$$p = \sqrt{\frac{g}{\delta}} = \sqrt{\frac{960EIg}{11Wl^3}}. \tag{p}$$

In calculating the effect of this mass on the frequency let us denote by $\dot{x}_{max}$ the maximum velocity of the vibrating body W. Then the maximum velocity of any point of the vertical bars at a distance ξ from the bottom is

$$(\dot{x}_1)_{max} = \dot{x}_{max}\frac{x_1}{\delta} = \dot{x}_{max}\frac{12}{11}\frac{\xi}{l}\left(1 - \frac{\xi^2}{l^2}\right) \tag{r}$$

and the maximum velocity at any point of the left portion of the horizontal bar CD

$$(\dot{x}_2)_{max} = \dot{x}_{max}\frac{x_2}{\delta} = \dot{x}_{max}\left[\frac{20}{11}\frac{\eta}{l}\left(3 - \frac{4\eta^2}{l^2}\right) - \frac{36}{11}\frac{\eta}{l}\left(1 - \frac{\eta}{l}\right)\right]. \tag{s}$$

The kinetic energy of the frame which must be added to the kinetic energy of the load W is

$$2\int_0^l \frac{w\dot{x}^2_{max}}{2g}\left(\frac{x_1}{\delta}\right)^2 d\xi + 2\int_0^{l/2} \frac{w\dot{x}^2_{max}}{2g}\left(\frac{x_2}{\delta}\right)^2 d\eta .$$

Substituting for the ratios x_1/δ and x_2/δ their expressions from (r) and (s) and integrating, the additional kinetic energy can be represented in the following form

$$\frac{w\alpha l}{2g}(\dot{x})^2_{max}$$

where α is a constant factor.

The equation for frequency of vibration now becomes

$$p = \sqrt{\frac{960EIg}{11(W + \alpha wl)l^3}}. \tag{t}$$

3. Determine the frequency of lateral vibrations of the frame shown in Fig. 60b.

Solution. The frequency of these vibrations, if the mass of the frame is neglected, can be calculated by using the formulas of problem 5, see p. 7. To take into account the mass of the frame, the bending of the frame must be considered. If x is the lateral displacement of the load W together with the horizontal bar CD, the horizontal displacement of any point of the vertical bars at a distance ξ from the bottom, from consideration of the bending of the frame, is

$$x_1 = x - \frac{x}{3}\left(1 - \frac{\xi}{l}\right) - \frac{2}{3}x\left[\frac{3}{2}\left(1 - \frac{\xi}{l}\right)^2 - \frac{1}{2}\left(1 - \frac{\xi}{l}\right)^3\right]. \tag{u}$$

The kinetic energy of the vertical bars is

$$2\int_0^l \frac{w\dot{x}_1^2}{2g}d\xi = \frac{\alpha wl}{g}\dot{x}^2,$$

where α is a constant factor which is obtained after substituting for x_1 its expression from (u) and integrating. In considering the kinetic energy of the horizontal bar we take into consideration only the horizontal component $\dot{x}$ of the velocities of the particles of the bar. Then the total kinetic energy of the load together with the frame is

$$\frac{W\dot{x}^2}{2g} + \frac{(1 + 2\alpha)wl\dot{x}^2}{2g},$$

and the frequency is obtained from the equation (see prob. 5, p. 7).

$$f = \frac{1}{2\pi}\sqrt{\frac{4EIg}{[W + (1 + 2\alpha)wl]l^3}}.$$

17. Critical Speed of a Rotating Shaft.—It is well known that rotating shafts at certain speeds become dynamically unstable and large vibrations are likely to develop. This phenomenon is due to resonance effects and a simple example will show that the *critical speed* for a shaft is that speed at which the number of revolutions per second of the shaft is equal to the frequency of its natural lateral vibration.*

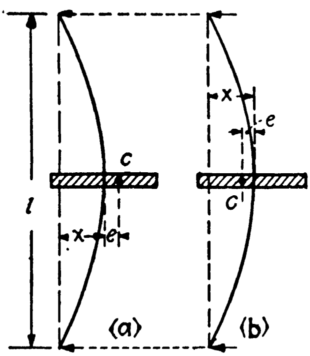

FIG. 61.

Shaft with One Disc.—In order to exclude from our consideration the effect of the weight of the shaft and so make the problem as simple as possible, a vertical shaft with one circular disc will be taken (Fig. 61, *a*). Let C be the center of gravity of the disc and e a small eccentricity, i.e., the distance of C from the axis of the shaft. During rotation, due to the eccentricity e, a centrifugal force will act on the shaft, and will produce deflection. The magnitude of the deflection x can easily be obtained from the condition of equilibrium of the centrifugal force and the reactive force P of the deflected shaft. This latter force is proportional to the deflection x, and can be represented in the following form,

$$P = kx.$$

The magnitude of the factor k can be calculated provided the dimensions of the shaft and the conditions at the supports be known. Assuming, for instance, that the shaft has a uniform section and the disc is in the middle between the supports, we have

$$k = \frac{48EI}{l^3}.$$

* A more detailed discussion of lateral vibrations of a shaft is given in Articles 39 and 49.

Now from the condition of equilibrium the following equation for determining x will be obtained

$$\frac{W}{g}(x + e)\omega^2 = kx, \tag{a}$$

in which W/g is the mass of the disc, ω is angular velocity of the shaft.

From eq. (a) we have,

$$x = \frac{e}{\dfrac{k}{\omega^2}\dfrac{g}{W} - 1}. \tag{b}$$

Remembering (see eq. (2), p. 2) that

$$\frac{kg}{W} = p^2,$$

it can be concluded from (b) that the deflection x tends to increase rapidly as ω approaches p, i.e., when the number of revolutions per second of the shaft approaches the frequency of the lateral vibrations of the shaft and disc. The critical value of the speed will be

$$\omega_{cr} = \sqrt{\frac{kg}{W}}. \tag{44}$$

At this speed the denominator of (b) becomes zero and large lateral vibrations in the shaft occur. It is interesting to note that at speeds higher than the critical quiet running conditions will again prevail. The experiments show that in this case the center of gravity C will be situated between the line joining the supports and the deflected axis of the shaft as shown in Fig. 61, b. The equation for determining the deflection will be

$$\frac{W}{g}(x - e)\omega^2 = kx,$$

from which

$$x = \frac{e}{1 - \dfrac{kg}{\omega^2 W}}. \tag{c}$$

It is seen that now with increasing ω the deflection x decreases and approaches the limit e, i.e., at very high speeds the center of gravity of the disc approaches the line joining the supports and the deflected shaft rotates about the center of gravity C.

Shaft Loaded with Several Discs.—It has been shown above in a simple example that the critical number of revolutions per second of a shaft is equal to the frequency of the natural lateral vibration of this shaft. Determining this frequency by using Rayleigh's method the critical speed for a shaft with many discs (Fig. 62) can easily be established. Let W_1, W_2, W_3 denote the loads and x_1, x_2, x_3 denote the corresponding statical deflections. Then the potential energy of deformation stored in the beam during bending will be

$$V = \frac{W_1x_1}{2} + \frac{W_2x_2}{2} + \frac{W_3x_3}{2}. \tag{d}$$

In calculating the period of the slowest type of vibration the static deflection curve shown in Fig. 62 can be taken as a good approximation for the deflection curve of the beam during vibration. The vertical displacements of the loads W_1, W_2 and W_3 during vibration can be written as:

$$x_1 \cos pt, \qquad x_2 \cos pt, \qquad x_3 \cos pt. \tag{e}$$

Then the maximum deflections of the shaft from the position of equilibrium are the same as those given in Fig. 62; therefore, the increase in the potential energy of the vibrating shaft during its deflection from the position of equilibrium to the extreme position will be given by equation (*d*). On the other hand the kinetic energy of the system becomes maximum at the moment when the shaft, during vibration, passes through its middle position. It will be noted, from eq. (*e*), that the velocities of the loads corresponding to this position are:

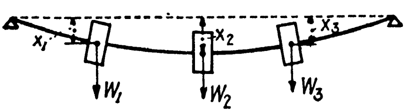

FIG. 62.

$$px_1, \qquad px_2, \qquad px_3$$

and the kinetic energy of the system becomes

$$\frac{p^2}{2g}(W_1x_1^2 + W_2x_2^2 + W_3x_3^2). \tag{f}$$

Equating (*d*) and (*f*), the following expression for p^2 will be obtained:

$$p^2 = \frac{g(W_1x_1 + W_2x_2 + W_3x_3)}{W_1x_1^2 + W_2x_2^2 + W_3x_3^2}. \tag{45}$$

The period of vibration is

$$\tau = \frac{2\pi}{p} = 2\pi\sqrt{\frac{W_1x_1^2 + W_2x_2^2 + W_3x_3^2}{g(W_1x_1 + W_2x_2 + W_3x_3)}}. \tag{46}$$

In general, when n loads are acting on the shaft the period of the lowest type of vibration will be

$$\tau = 2\pi\sqrt{\frac{\sum_1^n W_i x_i^2}{g\sum_1^n W_i x_i}}. \tag{47}$$

It is seen that for calculating τ the statical deflections x_1, x_2 $\cdots$ of the shaft alone are necessary. These quantities can easily be obtained by the usual methods. If the shaft has a variable cross section a graphical method for obtaining the deflections has to be used. The effect of the weight of the shaft itself also can be taken into account. It is necessary for this purpose to divide the shaft into several parts, the weights of which, applied to their respective centers of gravity, must be considered as concentrated loads.

Take, for instance, the shaft shown in Fig. 63, *a*, the diameters of which and the loads acting on it are shown in the figure. By constructing the polygon of forces (Fig. 63, *b*) and the corresponding funicular polygon (Fig. 63, *c*) the bending moment diagram will be obtained. In order to calculate the numerical value of the bending moment at any cross section of the shaft it is only necessary to measure the corresponding ordinate e of the moment diagram to the same scale as used for the length of the shaft and multiply it with the pole distance h measured to the scale of forces in the polygon of forces (in our case h = 80,000 lbs.). In order to obtain the deflection curve a construction of the second funicular polygon is necessary in which construction the bending moment diagram obtained above must be considered as an imaginary loading diagram. In order to take into account the variation in cross section of the shaft, the intensity of this imaginary loading at every section should be multiplied by I_0/I where I_0 = moment of inertia of the largest cross section of the shaft and I = moment of inertia of the portion of the shaft under consideration. In this manner the final imaginary loading represented by the shaded area (Fig. 63, *c*) is obtained. Subdividing this area into several parts, measuring the areas of these parts in square inches and multiplying them with the pole distance h measured in pounds, the imaginary loads measured in pounds-inches^2 will be obtained. For these loads,

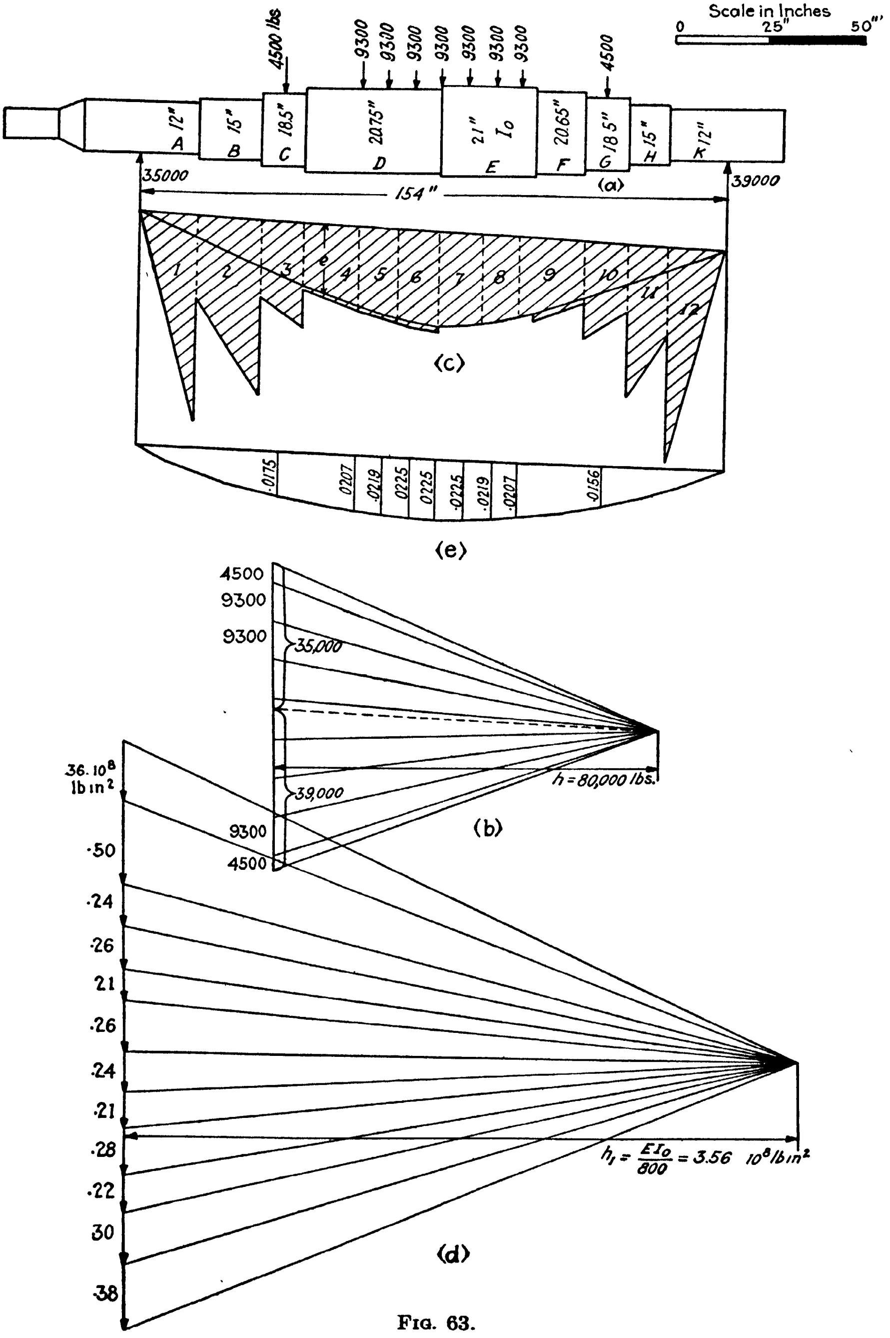
Scale in Inches
0
25"
50"
4500 lbs
9300
9300
9300
9300
9300
9300
9300
4500
12"
A
15"
B
18.5"
C
20.75"
D
21"
I_0
E
20.65"
F
18.5"
G
15"
H
12"
K
35000
39000
(a)
154"
1
2
3
e
4
5
6
7
8
9
10
11
12
(c)
.0175
.0207
.0219
.0225
.0225
.0225
.0219
.0207
.0156
(e)
4500
9300
9300
35,000
39,000
h = 80,000 lbs.
9300
4500
(b)
$36 \cdot 10^8$ lb in^2
·50
·24
·26
21
·26
·24
·21
·28
·22
30
·38
$h_1 = \frac{EI_0}{800} = 3.56 \; 10^8 \, lb \, in^2$
(d)

Fig. 63.

the second polygon of forces (Fig. 63, d) is constructed by taking a pole distance h_1 equal to EI_0/n where EI_0 is the largest flexural rigidity of the shaft and n is an integer (in our case $n = 800$). It should be noted that the imaginary loads and the pole distances EI/n have the same dimension, i.e., in.^{2}-lbs., and should be represented in the polygon of forces to the same scale. By using the second polygon of forces the second funicular polygon (Fig. 63, e) and the deflection curve of the shaft tangent to this polygon can easily be constructed. In order to get the numerical values of the deflections it is only necessary to measure them to the same scale to which the length of the shaft is drawn and divide them by the number n used above in the construction of the second polygon. All numerical results obtained from the drawing and necessary in using eq. (47) are given in the table below.

W lbs.	$x_i \times 10^2$ in.	$x_i^2 \times 10^4$ in.^{2}	Wx_i lbs. $\times$ in.	Wx_i^2 lbs. $\times$ in.^{2}
4500	1.75	3.05	79	1.37
9300	2.07	4.28	193	3.98
9300	2.19	4.80	204	4.47
9300	2 25	5.06	209	4.71
9300	2.25	5.06	209	4.71
9300	2.25	5.36	209	4.71
9300	2.19	4.88	204	4.47
9300	2.07	4 28	193	3.98
4500	1.56	2.43	70	1.09
			$\Sigma Wx_i = 1570$	$\Sigma Wx_i^2 = 33.09$

The critical number of revolutions per minute will be obtained now as follows:

$$N_{cr} = \frac{60}{\tau} = \frac{30}{\pi}\sqrt{\frac{g\sum\limits_{1}^{n} W_i x_i}{\sum\limits_{1}^{n} W_i x_i^2}} = \frac{30}{\pi}\sqrt{\frac{386 \times 1570}{33.09}} = 1290 \text{ R.P.M.}$$

It should be noted that the hubs of spiders or flywheels shrunk on the shaft increase the stiffness of the shaft and may raise its critical speed considerably. In considering this phenomenon it can be assumed that the stresses due to vibration are small and the shrink fit pressure between the hub and the shaft is sufficient to prevent any relative motion between these two parts, so that the hub can be considered as a portion of shaft of an enlarged diameter. Therefore the effect of the hub on the critical speed will be obtained by introducing this enlarged diameter in the graphical construction developed above.*

In the case of a grooved rotor (Fig. 64) if the distances between the grooves are of the same order as the depth of the groove, the material between two grooves does not take any bending stresses and the flexibility

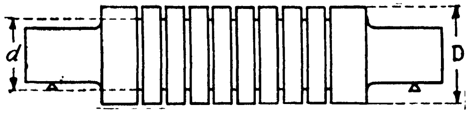

FIG. 64.

of such a rotor is near to one of the diameter d measured at the bottom of the grooves.†

It must be noted also that in Fig. 62 rigid supports were assumed. In certain cases the rigidity of the supports is small enough so as to produce a substantial effect on the magnitude of the critical speed. If the additional flexibility, due to deformation of the supports, is the same in a vertical and in a horizontal direction the effect of this flexibility can be easily taken into account. It is only necessary to add to the deflections x_1, x_2 and x_3 of the previous calculations the vertical displacement due to the deformation of the supports under the action of the loads W_1, W_2 and W_3. Such additional deflections will lower the critical speed of the shaft.‡

18. General Case of Disturbing Force.—In the previous discussion of forced vibrations (see articles 4 and 9) a particular case of a disturbing force proportional to sin ωt was considered. In general case a periodical disturbing force is a function of time $f(t)$ which can be represented in the form of a trigonometrical series such as

$$f(t) = a_0 + a_1 \cos \omega t + a_2 \cos 2\omega t + \ldots b_1 \sin \omega t + b_2 \sin 2\omega t + \ldots, \quad (a)$$

* Prof. A. Stodola in his book "Dampf- und Gas Turbinen," 6th ed. (1924), p. 383, gives an example where such a consideration of the stiffening effect of shrunk on parts gave a satisfactory result and the calculated critical speed was in good agreement with the experiment. See also paper by B. Eck, Versteifender Einfluss der Turbinenscheiben, V. D. I., Bd. 72, 1928, S. 51.

† B. Eck, loc. cit.

‡ The case when the rigidities of the supports in two perpendicular directions are different is discussed on p. 296.

in which

$$f_1 = \frac{\omega}{2\pi} \text{ is the frequency of the disturbing force,}$$

$$\tau_1 = \frac{2\pi}{\omega} \text{ is the period of the disturbing force.}$$

In order to calculate any one of the coefficients of eq. (a) provided $f(t)$ be known the following procedure must be followed. Assume that any coefficient a_i is desired, then both sides of the equation must be multiplied by $\cos i\omega t\, dt$ and integrated from $t = 0$ to $t = \tau_1$. It can be shown that

$$\int_0^{\tau_1} a_0 \cos i\omega t\, dt = 0; \qquad \int_0^{\tau_1} a_k \cos k\omega t \cos i\omega t\, dt = 0;$$

$$\int_0^{\tau_1} b_k \sin k\omega t \cos i\omega t\, dt = 0; \qquad \int_0^{\tau_1} a_i \cos^2 i\omega t\, dt = \frac{a_i}{2}\tau_1,$$

where i and k denote integer numbers 1, 2, 3, $\cdots$. By using these formulas we obtain, from eq. (a),

$$a_i = \frac{2}{\tau_1}\int_0^{\tau_1} f(t) \cos i\omega t\, dt. \tag{b}$$

In the same manner, by multiplying eq. (a) by $\sin i\omega t\, dt$, we obtain

$$b_i = \frac{2}{\tau_1}\int_0^{\tau_1} f(t) \sin i\omega t\, dt. \tag{c}$$

Finally, multiplying eq. (a) by dt and integrating from $t = 0$ to $t = \tau_1$, we have

$$a_0 = \frac{1}{\tau_1}\int_0^{\tau_1} f(t)\, dt. \tag{d}$$

It is seen that by using formulas (b), (c) and (d), the coefficients of eq. (a) can be calculated if $f(t)$ be known analytically. If $f(t)$ be given graphically, while no analytical expression is available, some approximate numerical method for calculating the integrals (b), (c) and (d) must be used or they can be obtained mechanically by using one of the instruments for analyzing curves in a trigonometrical series.*

* A discussion of various methods of analyzing curves in a trigonometrical series and a description of the instruments for harmonical analysis can be found in the book: "Practical Analysis," by H. von Sanden.

Assuming that the disturbing force is represented in the form of a trigonometrical series, the equation for forced vibrations will be (see eq. (32), p. 38).

$$\ddot{x} + 2n\dot{x} + p^2x = a_0 + a_1 \cos \omega t + a_2 \cos 2\omega t + \cdots$$
$$+ b_1 \sin \omega t + b_2 \sin 2\omega t + \cdots. \qquad (e)$$

The general solution of this equation will consist of two parts, one of free vibrations (see eq. (26), p. 33) and one of forced vibrations. The free vibrations will be gradually damped due to friction. In considering the forced vibration it must be noted that in the case of a linear equation, such as eq. (*e*), the forced vibrations will be obtained by superimposing the forced vibrations produced by every term of the series (*a*). These latter vibrations can be found in the same manner as explained in article (9) and on the basis of solution (35) (see p. 40) it can be concluded that large forced vibrations may occur when the period of one of the terms of series (*a*) coincides with the period τ of the natural vibrations of the system, i.e., if the period τ_1 of the disturbing force is equal to or a multiple of the period τ.

As an example consider vibrations produced in the frame *ABCD* by the inertia forces of a horizontal engine (Fig. 65) rotating with constant angular velocity ω. Assume that the horizontal beam *BC* is very rigid and that horizontal vibrations due to bending of the columns alone should be considered. The natural period of these vibrations can easily be obtained. It is only necessary to calculate the statical deflection δ_{st} of the top of the frame under the action of a horizontal force Q equal to the weight of the engine together with the weight of horizontal platform *BC*. (The mass of the vertical columns is neglected in this calculation.) Assuming that the beam *BC* is absolutely rigid and rests on two columns, we have

$$\delta_{st} = \frac{Q}{3EI}\left(\frac{h}{2}\right)^3.$$

Substituting this in the equation,

$$\tau = 2\pi\sqrt{\frac{\delta_{st}}{g}},$$

the period of natural vibration will be found.

In the case under consideration, forced vibrations will be produced by the inertia forces of the rotating and reciprocating masses of the engine.

In considering these forces the mass of the connecting rod can be replaced with sufficient accuracy by two masses, one at the crank pin and the second at the cross-head. To the same two points all other unbalanced masses in motion readily can be reduced, so that finally only two masses M_1 and M have to be taken into consideration (Fig. 65, *b*). The horizontal component of the inertia force of the mass M_1 is

$$-M_1\omega^2 r \cos \omega t, \tag{f}$$

in which ω is angular velocity of the engine,
r is the radius of the crank,
ωt is the angle of the crank to the horizontal axis.

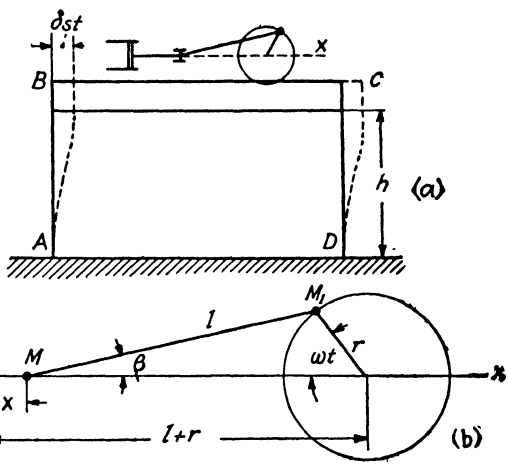

Fig. 65.

The motion of the reciprocating mass M is more complicated. Let x denote the displacement of M from the dead position and β, the angle between the connecting rod and the x axis. From the figure we have,

$$x = l(1 - \cos\beta) + r(1 - \cos\omega t) \tag{g}$$

and

$$r \sin\omega t = l \sin\beta. \tag{h}$$

From (h),

$$\sin\beta = \frac{r}{l}\sin\omega t.$$

The length l is usually several times larger than r so that with sufficient accuracy it can be assumed that

$$\cos \beta = \sqrt{1 - \frac{r^2}{l^2} \sin^2 \omega t} \approx 1 - \frac{r^2}{2l^2} \sin^2 \omega t.$$

Substituting in eq. (g),

$$x = r(1 - \cos \omega t) + \frac{r^2}{2l} \sin^2 \omega t. \qquad (k)$$

From this equation the velocity of the reciprocating masses will be

$$\dot{x} = r\omega \sin \omega t + \frac{r^2\omega}{2l} \sin 2\omega t$$

and the corresponding inertia forces will be

$$-M\ddot{x} = - M\omega^2 r \left(\cos \omega t + \frac{r}{l} \cos 2\omega t \right). \qquad (l)$$

Combining this with (f) the complete expression for the disturbing force will be obtained. It will be noted that this expression consists of two terms, one having a frequency equal to the number of revolutions of the machine and another having twice as high a frequency. From this it can be concluded that in the case under consideration we have two critical speeds of the engine: the first when the number of revolutions of the machine per second is equal to the frequency $1/\tau$ of the natural vibrations of the system and the second when the number of revolutions of the machine is half of the above value. By a suitable choice of the rigidity of the columns AB and CD it is always possible to ascertain conditions sufficiently far away from such critical speeds and to remove in this manner the possibility of large vibrations. It must be noted that the expression (l) for the inertia force of the reciprocating masses was obtained by making several approximations. A more accurate solution will also contain harmonics of a higher order. This means that there will be critical speeds of an order lower than those considered above, but usually these are of no practical importance because the corresponding forces are too small to produce substantial vibrations of the system.

In the above consideration the transient condition was excluded. It was assumed that the free vibrations of the system, usually generated at the beginning of the motion, have been damped out by friction and forced vibrations alone are being considered. When the displacement of a system at the beginning of the motion is to be investigated or when the acting force cannot be accurately represented by few terms of series (a) another way of calculating displacements of a vibrating system, based on solution (7) (see p. 4) of the equation of free harmonic vibration, has certain advantages. To explain the method let us consider the system shown in Fig. 1. We assume that at the initial instant ($t = 0$) the body is at rest in its position of statical equilibrium. A vertical disturbing force of the magnitude q per unit mass of the body W is applied at the initial instant and it is required to find the displacement of the body at any instant $t = t_1$. The variation of the force with time is represented by the curve MN in Fig. 66. To calculate the required displacement we imagine the continuous action of the force divided into small intervals dt.* The impulse qdt of the force during one of these elemental intervals is shown in Fig. 66 by the shaded strip. Let us now calculate the displacement of the body at the instant t_1 produced by this elemental impulse. As a result of this impulse an increase in the velocity of the body will be generated at the instant t. The magnitude of the velocity increase is found from the equation

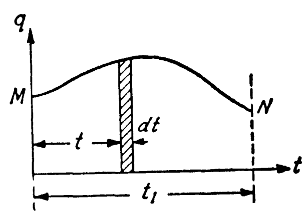

Fig. 66.

$$\frac{d\dot{x}}{dt} = q,$$

from which

$$d\dot{x} = qdt. \qquad (a)$$

The displacement of the body at the instant t_1 corresponding to the velocity $d\dot{x}$ which was communicated to it at the instant t may be calculated by the use of solution (7). It is seen from this solution that by reason of the initial velocity $\dot{x}_0$ the displacement at any instant t is $(\dot{x}_0 \sin pt)/p$. Hence the velocity $d\dot{x}$ communicated at the instant t to the body produces a displacement of the body at the instant t_1 given by

$$dx = \frac{qdt}{p} \sin p(t_1 - t). \qquad (b)$$

This is the displacement due to one elemental impulse only. In order to

* This method has been used by Lord Rayleigh, see "Theory of Sound," Vol. 1, p. 74, 1894. See also the book by G. Duffing, "Erzwungene Schwingungen," 1918, p. 14, and the book "Theoretical Mechanics," by L. Loiziansky and A. Lurje, vol. 3, p. 338, 1934, Moscow.

obtain the total displacement of the body produced by the continuous action of the force q, it is necessary to make a summation of all the elemental displacements given by expression (b). The summation yields:

$$x = \frac{1}{p}\int_0^{t_1} q \sin p(t_1 - t)dt. \tag{48}$$

This expression represents the complete displacement produced by the force q acting during the interval from $t = 0$ to $t = t_1$. It includes both forced and free vibrations and may become useful in studying the motion of the system at starting. It can be used also in cases where an analytical expression for the disturbing force is not known and where the force q is given graphically or numerically. It is only necessary in such a case to determine the magnitude of the integral (48) by using one of the approximate methods of integration.*

As an example of the application of this method, vibration under the action of a disturbing force $q = u \sin \omega t$ will now be considered. Substituting this expression of q in eq. (48) and observing that

$$\sin \omega t \sin p(t - t_1) = \tfrac{1}{2}\{\cos(\omega t + pt - pt_1) - \cos(\omega t - pt + pt_1)\}$$

we obtain

$$x = \frac{u}{p^2 - \omega^2}\left(\sin \omega t_1 - \frac{\omega}{p}\sin pt_1\right)$$

which coincides with solution (21) for $t = t_1$.

Equation (48) can be used also in cases where it is necessary to find the displacement of the load W (see Fig. 1) resulting from several impulses. Assume, for instance, that due to impulses obtained by the load W at the moments $t', t'', t''', \cdots$ increments of the speed $\Delta_1\dot{x}, \Delta_2\dot{x}, \Delta_3\dot{x}, \cdots$ be produced. Then from equations (b) and (48) the displacement at any moment t_1 will be,

$$x = \frac{1}{p}\left[\Delta_1\dot{x}\sin p(t_1 - t') + \Delta_2\dot{x}\sin p(t_1 - t'') + \Delta_3\dot{x}\sin p(t_1 - t''') + \cdots\right].$$

This displacement can be obtained very easily graphically by considering $\Delta_1\dot{x}, \Delta_2\dot{x}, \cdots$ as vectors inclined to the horizontal axis at angles $p(t_1 - t')$, $p(t_1 - t'')$, $\cdots$ Fig. (67). The vertical projection OC_1 of the geometrical sum OC of these vectors, divided by p, will then represent the displacement x given by the above equation.

* See von Sanden, "Practical Analysis," London, 1924.

In cases where a constant force q is applied at the moment $t = 0$ to the load W (Fig. 1) the displacement of the load at any moment t_1 becomes from eq. (48):

$$x = \frac{q}{p}\int_0^{t_1} \sin p(t_1 - t)dt = \frac{q}{p^2}(1 - \cos pt_1), \tag{d}$$

where q/p^2 is statical deflection due to the force q (see p. 14). It is seen, from (d), that the maximum deflection during vibrations produced by a suddenly applied force is equal to twice the statical deflection corresponding to the same force.

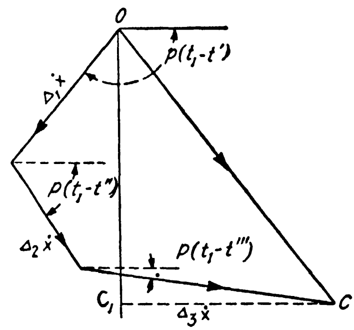

FIG. 67.

It was assumed that the suddenly applied, constant force q is acting all time from $t = 0$ to $t = t_1$. If the force q acts only during a certain interval Δ of that time and then is suddenly removed, the motion of the body, after removal of the force, can also be obtained from eq. 48. We write this equation in the following form

$$x = \frac{1}{p}\int_0^{\Delta} q \sin p(t_1 - t)dt + \frac{1}{p}\int_{\Delta}^{t_1} q \sin p(t_1 - t)\, dt.$$

Observing that q is zero for $\Delta < t < t_1$, the second integral on the right side vanishes and we obtain

$$x = \frac{1}{p}\int_0^{\Delta} q \sin p(t_1 - t)dt = \frac{q}{p^2}\Big|\cos p(t_1 - t)\Big|_0^{\Delta}$$

$$= \frac{q}{p^2}\Big[\cos p(t_1 - \Delta) - \cos pt_1\Big] = \frac{2q}{p^2}\sin\frac{p\Delta}{2}\sin p\left(t_1 - \frac{\Delta}{2}\right). \tag{e}$$

Thus a constant force acting during an interval of time Δ produces a simple sinusoidal motion of the amplitude which depends on the ratio of the interval Δ to the period $\tau = 2\pi/p$ of the free vibration of the system. Taking, for instance, $\Delta/\tau = \frac{1}{2}$ we find $\sin (p\Delta/2) = 1$ and the amplitude of vibration (e) is twice as large as the statical deflection q/p^2. If we take $\Delta = \tau$, $\sin (p\Delta/2) = 0$ and there will be no vibration at all after removal of the force. Considering the system in Fig. 1 we have in the first case the force q removed when the weight W is in its lowest position. In the second case the force is removed when the body is in its highest position, which is its position of static equilibrium.

If the loading and unloading of the system is repeated several times and τ_1 is the constant interval of time between two consecutive applications of the force, the resulting motion is

$$x = \frac{2q}{p^2}\sin\frac{p\Delta}{2}\left[\sin p\left(t_1 - \frac{\Delta}{2}\right) + \sin p\left(t_1 - \tau_1 - \frac{\Delta}{2}\right) + \sin p\left(t_1 - 2\tau_1 - \frac{\Delta}{2}\right) + \cdots\right].$$

We see that by taking $\tau_1 = 2\pi/p$ the phenomenon of resonance takes place and the amplitude of vibration will be gradually built up.

It was assumed in the derivation of eq. (48) that the system is at rest initially. If there is some initial displacement x_0 and an initial velocity $\dot{x}_0$, the total displacement at an instant t_1 will be obtained by superposing on the displacement given by expression (48) the displacement due to the initial conditions. In this case we obtain

$$x = x_0 \cos pt + \frac{\dot{x}_0}{p}\sin pt + \frac{1}{p}\int_0^{t_1} q \sin p(t_1 - t)dt. \tag{49}$$

If there is a viscous damping a similar method can be used in studying forced vibrations. From solution (30) we see that an initial velocity $\dot{x}_0$ produces a displacement of the body (Fig. 1) at an instant t which is given by

$$\frac{1}{p_1}\dot{x}_0\, e^{-nt}\sin p_1 t. \tag{e}$$

The quantity n defines the damping and $p_1 = \sqrt{p^2 - n^2}$. From this we conclude that a velocity $d\dot{x} = q\,dt$ communicated at an instant t produces a displacement at the instant t_1 equal to

$$dx = \frac{q}{p_1}e^{-n(t_1-t)}\sin p_1(t_1 - t)dt. \tag{f}$$

The complete displacement of the body resulting from the action of the force q from $t = 0$ to $t = t_1$, will be obtained by a summation of expressions (f). Thus we have

$$x = \frac{1}{p_1}\int_0^{t_1} qe^{-n(t_1-t)}\sin p_1(t_1 - t)dt. \tag{50}$$

This formula is useful in calculating displacements when the force q is given graphically or if it cannot be represented accurately by a few terms of the series (a).

19. Effect of Low Spots on Deflection of Rails.—As an example of an application of eq. (48) of the previous article let us consider the effect of low spots on deflection of rails. Due to the presence of a low spot on the rail some vertical displacement of a rolling wheel occurs which results in an additional vertical pressure on the rail. This additional pressure depends on the velocity of rolling and on the profile of the low spot. Taking the coordinate axis as shown in Fig. 68 we denote by l the length of the low

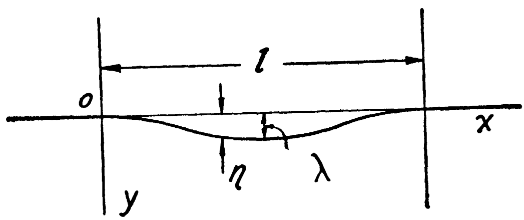

FIG. 68.

spot and by η the variable depth of the spot. The rail we consider as a beam on a uniform elastic foundation and we denote by k the concentrated vertical pressure which is required to produce a vertical deflection of the rail equal to one inch. If W denotes the weight of the wheel together with the weights of other parts rigidly connected with the wheel, the static deflection of the rail under the action of this weight is

$$\delta_{st} = \frac{W}{k}. \tag{a}$$

If the rail be considered an elastic spring, period of the free vibration of the wheel supported by the rail will be

$$\tau = 2\pi\sqrt{\frac{\delta_{st}}{g}}. \tag{b}$$

For a 100 lb. rail, with $EI = 44 \times 30 \times 10^6$ lb.in.^{2}, and with $W = 3000$ lb., we will find, for a usual rigidity of the track, that the wheel performs about 20 oscillations per second. Since this frequency is large in comparison with the frequency of oscillation of a locomotive cab on its springs, we can assume that the vibrations of the wheel are not transmitted to the cab and that the vertical pressure of the springs on the axle remains constant and equal to the spring borne weight. Let us now consider the forced vibrations of the wheel due to the low spot. We denote the dynamic deflection of the rail under the wheel by y during this vibration.* Then the vertical displacement of the

* This deflection is measured from the position of static equilibrium which the wheel has under the action of the weight W and of the spring borne weight.

wheel traveling along the spot of variable depth η is $y + \eta$ and the vertical inertia force of the wheel will be

$$-\frac{W}{g}\frac{d^2(y+\eta)}{dt^2}.$$

The reaction of the rail is $-ky$ and the equation of motion of the wheel in the vertical direction becomes:

$$\frac{W}{g}\frac{d^2(y+\eta)}{dt^2} + ky = 0,$$

from which

$$\frac{W}{g}\frac{d^2y}{dt^2} + ky = -\frac{W}{g}\frac{d^2\eta}{dt^2}. \tag{c}$$

If the shape of the low spot and the speed of the locomotive are known, the depth η and consequently the right side of eq. (c) can be expressed as functions of time. Thus we obtain the equation of forced vibration of the wheel produced by the low spot.

Let us consider a case when the shape of the low spot (Fig. 68) is given by the equation

$$\eta = \frac{\lambda}{2}\left(1 - \cos\frac{2\pi x}{l}\right), \tag{d}$$

in which λ denotes the depth of the low spot at the middle of its length.

If we begin to reckon time from the instant when the point of contact of the wheel and the rail coincides with the beginning of the low spot, Fig. 68, and if we denote the speed of the locomotive by v, we have $x = vt$, and we find, from eq. (d), that

$$\eta = \frac{\lambda}{2}\left(1 - \cos\frac{2\pi vt}{l}\right). \tag{e}$$

Substituting this into eq. (c) we obtain

$$\frac{W}{g}\frac{d^2y}{dt^2} + ky = -\frac{W}{g}\frac{\lambda}{2}\frac{4\pi^2v^2}{l^2}\cos\frac{2\pi vt}{l}.$$

Dividing by W/g, and using our previous notations this becomes:

$$\ddot{y} + p^2y = -\frac{2\lambda\pi^2v^2}{l^2}\cos\frac{2\pi vt}{l}. \tag{f}$$

If the right side of this equation be substituted into equation (48) of the previous article we find that the additional deflection of the rail caused by the dynamical effect of the low spot is

$$y = -\frac{2\pi^2\lambda v^2}{pl^2}\int_0^{t_1}\cos\frac{2\pi vt}{l}\sin p(t_1 - t)dt. \tag{g}$$

Performing the integration and denoting by τ_1 the time l/v required for the wheel to pass over the low spot, we obtain

$$y = \frac{\lambda}{2(1 - \tau_1^2/\tau^2)}\left(\cos\frac{2\pi t_1}{\tau_1} - \cos\frac{2\pi t_1}{\tau}\right). \tag{h}$$

It is seen that the additional deflection of the rail, produced by the low spot, is proportional to the depth λ of the spot and depends also on the ratio τ_1/τ. As the wheel is traveling along the low spot, the variation of the additional deflection is represented for several values of the ratio τ_1/τ by the curves in Fig. 69. The abscissas give the position of the wheel along the low spot, and the ordinates give the additional deflection expressed in terms of λ. As soon as the wheel enters the low spot the pressure on the rail and consequently the deflection of the rail begin to diminish (y is negative) while the wheel begins to accelerate in a downward direction. Then follows a retardation of this movement with corresponding increases in pressure and in deflection. From the figure we see that for $\tau_1 < \tau$ the maximum pressure occurs when the wheel is approaching the

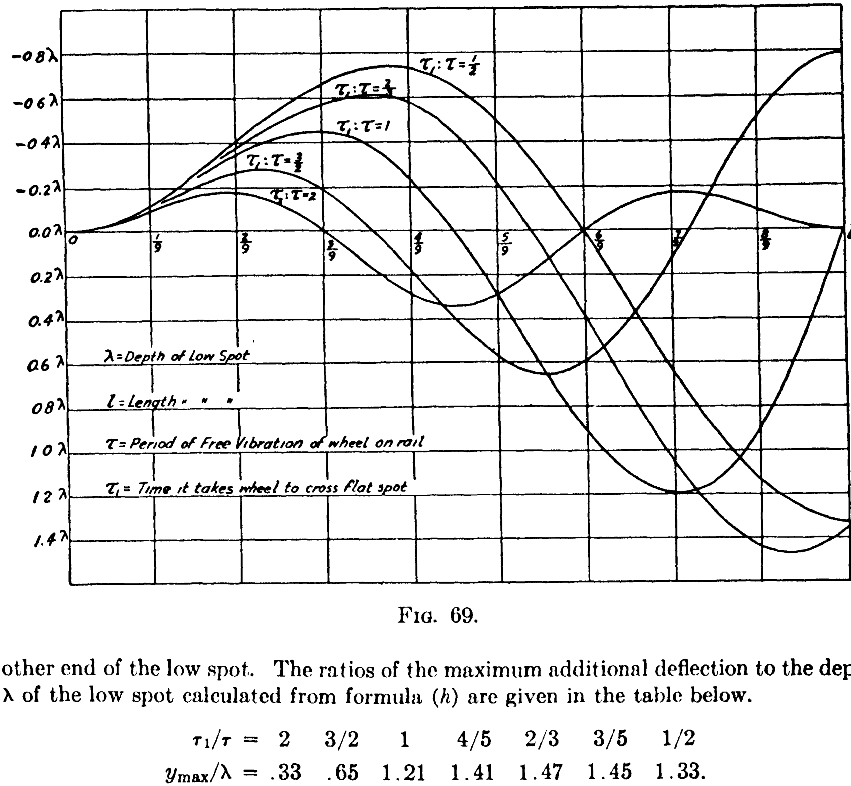

Fig. 69.

other end of the low spot. The ratios of the maximum additional deflection to the depth λ of the low spot calculated from formula (h) are given in the table below.

$\tau_1/\tau =$	2	3/2	1	4/5	2/3	3/5	1/2
$y_{max}/\lambda =$	.33	.65	1.21	1.41	1.47	1.45	1.33.

It is seen that the maximum value is about equal to 1.47. This ratio occurs when the speed of the locomotive is such that $(\tau_1/\tau) \approx 2/3$.

Similar calculations can be readily made if some other expression than eq. (e) is taken for the shape of the low spot provided that the assumed curve is tangent to the rail surface at the ends of the spot. If this condition is not fulfilled an impact at the ends of the low spot must be considered.*

* See author's papers in Transactions of the Institute of Engineers of Ways of Communication, 1915, S. Petersburg and in "Le Genie Civil," 1921, p. 551. See also Doctor Dissertation by B. K. Hovey, Gottingen, 1933.

In the discussion given above the mass of the vibrating part of the rail was neglected in comparison with the mass of the wheel. The error involved in this simplification of the problem is small if the time τ_1 required for the wheel to pass over the spot is long enough in comparison with the period of vibration of the rail on its elastic foundation. If it be assumed that the deflection of the rail under the action of its own weight is .002 in., the period of natural vibration of the rail moving in a vertical direction is $2\pi/\sqrt{500g}$ .0144 sec. This means that the solution (h) will give satisfactory results if $\tau_1 > .03$ sec.*

20. Self-Excited Vibration.—In discussing various problems of forced vibration we always assumed that the force producing vibration is independent of the vibratory motion. There are cases, however, in which a steady forced vibration is sustained by forces created by the vibratory motion itself and disappearing when the motion stops. Such vibrations are called *self-excited* or *self-induced* vibrations. In most musical instruments vibrations producing sound are of this kind. There are cases in engineering where self-excited vibrations are causing troubles.†

Vibration caused by friction. Vibration of a violin string under the action of the bow is a familiar case of self-excited vibration. The ability of the bow to maintain a steady vibration of the string depends on the fact that the coefficient of solid friction is not constant and diminishes as the velocity increases (Fig. 22, p. 31). During the vibration of the string acted upon by the bow the frictional force at the surface of contact does not remain constant. It is greater when the vibratory motion of the string is in the same direction as the motion of the bow, since the relative velocity of the string and bow is smaller under such condition than when the motion of the string is reversed. If one cycle of the string vibration be considered it may be seen‡ that during the half cycle in which the directions of motion of the string and of the bow coincide the friction force produces positive work on the string. During the second half of the cycle the work produced is negative. Observing that during the first half cycle the acting force is larger than during the second half, we may conclude that during a whole cycle positive work is produced with the result that forced vibration of the string will be built up. This forced vibration has the same frequency as the frequency of the natural vibration of the string.

* Recent experiments produced on Pennsylvania R. R. are in a satisfactory agreement with the figures given above for the ratio y_{max}/λ.

† Several cases of such vibrations are described and explained in a paper by J. G. Baker, Trans. Am. Soc. Mech. Engrs., vol. 55, 1933, and also in J. P. Den Hartog's paper, Proc. Fourth Intern. Congress Applied Mechanics, p. 36, 1934.

‡ It is assumed that the velocity of the bow is always greater than the velocity of the vibrating string.

The same type of vibration can be demonstrated by using the device shown in Fig. 36. In our previous discussion (see p. 57) it was assumed that the Coulomb friction remains constant, and it was found that in such a case the bar of the device will perform a simple harmonic motion. The experiments show, however, that the amplitude of vibration does not remain constant but grows with time. The explanation of this phenomenon is the same as in the previous case. Owing to a difference in relative velocity of the bar with respect to two discs the corresponding coefficients of friction are also different with the result that during each cycle positive work is produced on the bar. This work manifests itself in a gradual building up of the amplitude of vibration.

One of the earliest experiments with self-excited mechanical vibration was made by W. Froude,* who found that the vibrations of a pendulum swinging from a shaft, Fig. 70, might be maintained or even increased by rotating the shaft. Again the cause of this phenomenon is the solid friction acting upon the pendulum. If the direction of rotation of the shaft is as shown in the figure, the friction force is larger when the pendulum is moving to the right than for the reversed motion. Hence during each complete cycle positive work on the pendulum will be produced. It is obvious that the devices of Fig. 36 and Fig. 70 will demonstrate self-excited vibrations only as long as we have solid friction. In the case of viscous friction, the friction force increases with the velocity so that instead of exciting vibrations, it will gradually damp them out.

FIG. 70.

An example of self excited vibration has been experienced with a vertical machine, Fig. 71, consisting of a mass A driven by a motor B. There is considerable clearance between the shaft and the guide C, and the shaft can be considered a cantilever built in at the bottom and loaded at the top. The frequency of the natural lateral vibration of the shaft, which is also its critical or whirling speed, can be readily calculated in the usual way (see Art. 17). Experience shows that the machine is running smoothly as long as the shaft remains straight and does not touch the guide, but if for one reason or another the shaft strikes the guide, a violent whirling starts and is maintained indefinitely. This type of whirling may occur at any speed of the shaft, and it has the same frequency as the critical speed or frequency of the shaft mentioned above. In order to explain this type of whirling, let us consider the horizontal cross sections

* Lord Rayleigh, Theory of Sound, vol. 1, p. 212, 1894.

of the shaft and of the guide represented in Fig. 71, *b*. As soon as the shaft touches the guide a solid friction force F will be exerted on the shaft which tends to displace the shaft and thereby produces the whirl in the direction opposite to the rotation of the shaft. The pressure necessary for the existence of a friction force is provided by the centrifugal force of the mass A acting through the shaft against the guide.

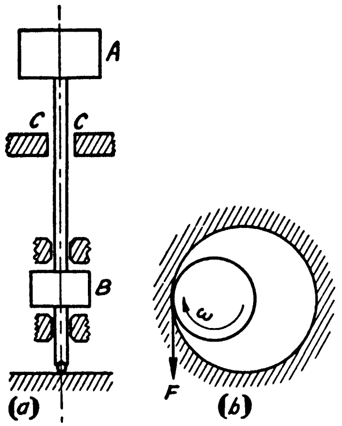

Fig. 71.

Vibration of Electric Transmission Lines. A wire stretched between two towers at a considerable distance apart, say about 300 ft., may, under certain conditions, vibrate violently at a low frequency, say 1 cycle per second. It happens usually when a rather strong transverse wind is blowing and the temperature is around 32° F., i.e., when the weather is favorable for formation of sleet on the wire. This phenomenon can be considered as a self-excited vibration.* If a transverse wind is blowing on a wire of a circular cross section (Fig. 72, *a*), the force exerted on the wire has the same direction as the wind. But in the case of an elongated cross section resulting from sleet formation (Fig. 72, *b*), the condition is different and the force acting on the wire has usually a direction different from that of the wind. A familiar example of this occurs on an aeroplane wing on which not only a *drag* in the direction of the wind but also a *lift*

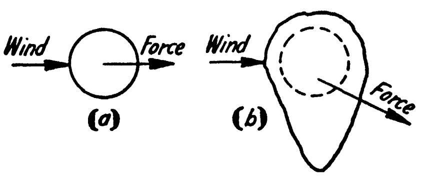

Fig. 72.

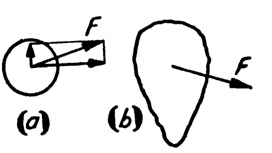

Fig. 73.

in a perpendicular direction are exerted. Let us now assume a vibration of the wire and consider the half cycle when the wire is moving downwards. In the case of a circular wire we shall have, owing to this motion, some air pressure in an upward direction. This force together with the horizontal wind pressure give an inclined force F (Fig. 73, *a*), which has an

* J. P. Den Hartog, Trans. Am. Inst. El. Engrs., 1932, p. 1074.

upward component opposing the motion of the wire. Thus we have a damping action which will arrest the vibration. In the case of an elongated cross section (Fig. 73, *b*) it may happen, as it was explained above, that due to the action of horizontal wind together with downward motion of the wire a force F having a component in a downward direction may be exerted on the wire so that it produces positive work during the downward motion of the wire. During the second half of the cycle, when the wire is moving upwards, the direction of the air pressure due to wire motion changes sign so that the combined effect of this pressure and the horizontal wind may produce a force with vertical component directed upwards. Thus again we have positive work produced during the motion of the wire resulting in a building up of vibrations.

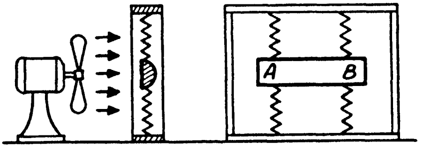

Fig. 74.

The above type of vibration can be demonstrated by using a device shown in Fig. 74. A light wooden bar suspended on flexible springs and with its flat side turned perpendicular to the wind of a fan, may be brought into violent vibrations in a vertical plane. The explanation of this vibration follows from the fact that a semicircular cross section satisfied the condition discussed above, so that the combined effect of the wind and of the vertical motion of the bar results in a force on the bar having always a vertical component in the direction of the vertical motion. Thus positive work is produced during the vibration.

CHAPTER II

VIBRATION OF SYSTEMS WITH NON-LINEAR CHARACTERISTICS

21. Examples of Non-Linear Systems.—In discussing vibration problems of the previous chapter it was always assumed that the deformation of a spring follows Hooke's law, i.e., the force in a spring is proportional to the deformation. It was assumed also that in the case of damping the resisting force is a linear function of the velocity of motion. As a result of these assumptions we always had vibrations of a system represented by a linear differential equation with constant coefficients. There are many practical problems in which these assumptions represent satisfactory actual conditions, however there are also systems in which a linear differential equation with constant coefficients is no longer sufficient to describe the actual motion so that a general investigation of vibrations requires a discussion of non-linear differential equations. Such systems are called *systems with non-linear characteristics.* One kind of such systems we have when the restoring force of a spring is not proportional to the displacement of the system from its position of equilibrium.

Sometimes, for instance, an organic material such as rubber or leather is used in couplings and vibrations absorbers. The tensile test diagram for these materials has the shape shown in Fig. 75; thus the modulus of elasticity increases with the elongation. For small amplitudes of vibration this variation in modulus may be negligible but with increasing amplitude the increase in modulus may result in a substantial increase in the frequency of vibration.

Another example of variable flexibility is met with in the case of structures made of such materials as cast iron or concrete. In both cases the tensile test diagram has the shape, shown in Fig. 76, i.e., the modulus of elasticity decreases with the deformation. Therefore some decrease in the frequency with increase of amplitude of vibration must be expected.

Sometimes special types of steel springs are used, such that their elastic characteristics vary with the displacement. The natural frequency of systems involving such springs depends on the magnitude of amplitude. By using such types of springs the unfavorable effect of resonance can be diminished. If, due to resonance, the amplitude of vibration begins to increase the frequency of the vibration changes, i.e., the resonance condition disappears. A simple example of such a spring is shown in Fig. 77. The flat spring, supporting the weight W, is built in

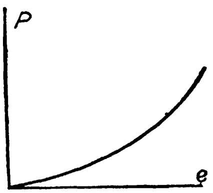

FIG. 75.

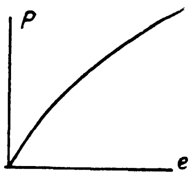

FIG. 76.

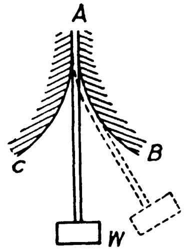

FIG. 77.

at the end A. During vibration the spring is partially in contact with one of two cylindrical surfaces AB or AC. Due to this fact the free length of the cantilever varies with the amplitude so that the rigidity of the spring increases with increasing deflection. The conditions are the same as in the case represented in Fig. 75, i.e., the frequency of vibration increases with an increase in amplitude.

If the dimensions of the spring and the shape of the curves AB and AC are known, a curve representing the restoring force as a function of the deflection of the end of the spring can easily be obtained.

As another example of non-linear system is the vibration along the x axis of a mass m attached to a stretched wire AB (Fig. 78). Assume

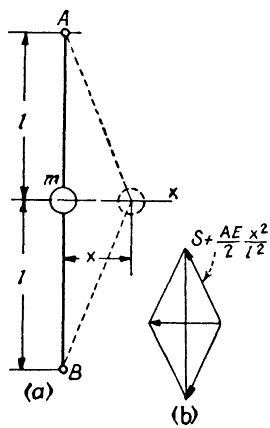

FIG. 78.

- S is initial tensile force in the wire,
- x is small displacement of the mass m in a horizontal direction,
- A is cross sectional area of the wire,
- E is modulus of elasticity of the wire.

The unit elongation of the wire, due to a displacement x, is

$$\frac{\sqrt{l^2 + x^2} - l}{l} \approx \frac{x^2}{2l^2}.$$

The corresponding tensile force in the wire is

$$S + AE\frac{x^2}{2l^2},$$

and the restoring force acting on the mass m (Fig. 78, b) will be

$$\left(S + AE\frac{x^2}{2l^2}\right)\frac{2x}{\sqrt{l^2 + x^2}} \approx \frac{2Sx}{l} + AE\frac{x^3}{l^3}.$$

The differential equation of motion of the mass m thus becomes

$$m\ddot{x} + \frac{2Sx}{l} + AE\frac{x^3}{l^3} = 0. \qquad (a)$$

It is seen that in the case of very small displacements and when the initial tensile force S is sufficiently large the last term on the left side of eq. (a) can be neglected and a simple harmonic vibration of the mass m in a horizontal direction will be obtained. Otherwise, all three terms of eq. (a) must be taken into consideration. In such a case the restoring force will increase in greater proportion than the displacement and the frequency of vibration will increase with the amplitude.

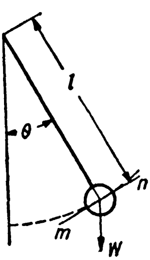

FIG. 79.

In the case of a simple mathematical pendulum (Fig. 79) by applying d'Alembert's principle and by projecting the weight W and the inertia force on the direction of the tangent mn the following equation of motion will be obtained:

$$\frac{Wl}{g}\ddot{\theta} + W\sin\theta = 0$$

or

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0, \qquad (b)$$

in which l is length of the pendulum, and θ is angle between the pendulum and the vertical.

It is seen that only in the case of small amplitudes, when $\sin\theta \approx \theta$, the oscillations of such a pendulum can be considered as simple harmonic. If the amplitudes are not small a more complicated motion takes place and the period of oscillation will depend on the magnitude of the amplitude. It is clear that the restoring force is not proportional to the displacement but increases at a lesser rate so that the frequency will decrease with an increase in amplitude of vibration. Expanding $\sin\theta$ in a power series and taking only the two first terms of the series, the following equation, instead of eq. (*b*), will be obtained

$$\ddot{\theta} + \frac{g}{l}(\theta - \theta^3/6) = 0. \tag{c}$$

Comparing this equation with eq. (*a*) it is easy to see that the non-linear terms have opposite signs. Hence by combining the pendulum with a horizontal stretched string (Fig. 80) attached to the bar of the pendulum at B and perpendicular to the plane of oscillation, a better approximation to isochronic oscillations may be obtained.

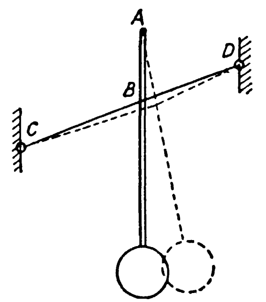

FIG. 80.

In Fig. 81 another example is given of a system in which the period of vibration depends on the amplitude. A mass m performs vibrations between two springs by sliding without friction along the bar AB. Measuring the displacements from the middle position of the mass m the variation of the restoring force with the displacement can be represented graphically as shown in Fig. 82. The frequency of the vibrations will depend not only on the spring constant but also on the magnitude of the clearance a and on the initial conditions. Assume, for instance, that

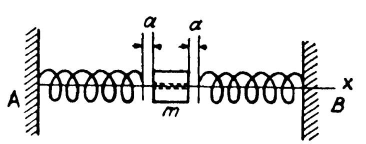

FIG. 81.

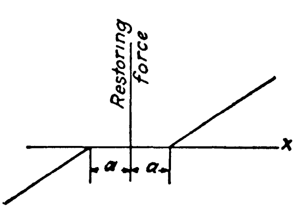

FIG. 82.

at the initial moment ($t = 0$) the mass m is in its middle position and has an initial velocity v in the x direction. Then the time necessary to cross the clearance a will be

$$t_1 = \frac{a}{v}. \tag{d}$$

After crossing the clearance, the mass m comes in contact with the spring and the further motion in the x direction will be simple harmonic. The time during which the velocity of the mass is changing from v to 0 (quarter period of the simple harmonic motion) will be (see eq. (5), p. 3)

$$t_2 = \frac{\pi}{2}\sqrt{\frac{m}{k}}, \tag{e}$$

where k is spring constant. The complete period of vibration of the mass m is

$$\tau = 4(t_1 + t_2) = \frac{4a}{v} + 2\pi\sqrt{\frac{m}{k}}. \tag{f}$$

For a given magnitude of clearance, a given mass m and a given spring constant k the period of vibration depends only on the initial velocity v. The period becomes very large for small values of v and decreases with increase of v, approaching the limit $\tau_0 = 2\pi\sqrt{m/k}$ (see Fig. 83) when $v = \infty$. Such conditions always are obtained if there are clearances in the system between the vibrating mass and the spring.

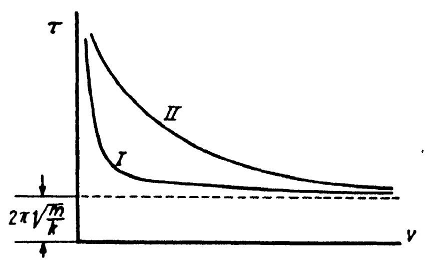

FIG. 83.

If the clearances are very small, the period τ remains practically constant for the larger part of the range of the speed v, as shown in Fig. 83 by curve 1. With increase in clearance for a considerable part of the range of speed v a pronounced variation in period of vibration takes place (curve II in Fig. 83). The period of vibration of such a system may have any value between $\tau = \infty$ and $\tau = \tau_0$. If a periodic disturbing force, having a period larger than τ_0, is acting, it will always be possible to give to the mass m such an impulse that the corresponding period of vibration will become equal to τ and in such manner resonance conditions will be established. Some heavy vibrations in electric locomotives have been explained in this manner.*

Another kind of non-linear systems we have when the damping forces are not represented by a linear function of the velocity. For instance, the resistance of air or of liquid, at considerable speed, can be taken proportional to the square of the velocity and the equation for the vibratory motion of a body in such a resisting medium will no longer be a linear one, although the spring of the system may follow Hooke's law.

* See A. Wichert, "Schüttelerscheinungen in elektrischen Lokomotiven," Forschungsarbeiten, No. 277, 1924, Berlin.

22. Vibrations of Systems with Non-linear Restoring Force.—If damping be neglected the general equation of motion in this case has the form

$$\frac{W}{g}\ddot{x} + k^2 f(x) = 0 \tag{a}$$

or

$$\ddot{x} + p^2 f(x) = 0, \tag{51}$$

in which $p^2 f(x)$ represents the restoring force per unit mass as a function of the displacement x. In order to get the first integral of eq. (51) we multiply it by dx/dt, then it can be represented in the following form:

$$\frac{dx}{dt} d\left(\frac{dx}{dt}\right) + p^2 f(x)dx = 0$$

or

$$\frac{1}{2} d\left(\frac{dx}{dt}\right)^2 + p^2 f(x)dx = 0,$$

from which, by integration we obtain

$$1/2\left(\frac{dx}{dt}\right)^2 + p^2 \int f(x)dx = 0. \tag{b}$$

If $f(x)$ and the initial conditions are known, the velocity of motion for any position of the system can be calculated from eq. (b). Assume, for instance, that the variation in the restoring force with the displacement is given by curve Om (see Fig. 84) and that in the initial moment $t = 0$, the system has a displacement equal to x_0 and an initial velocity equal to zero. Then, from eq. (b), for any position of the system we have

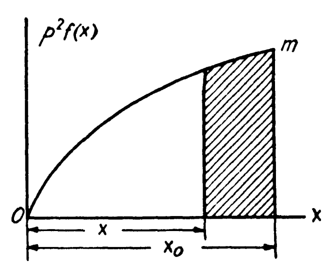

FIG. 84.

$$1/2\left(\frac{dx}{dt}\right)^2 = p^2 \int_x^{x_0} f(x)dx, \tag{c}$$

which means that at any position of the system the kinetic energy is equal to the difference of the potential energy which was stored in the spring in the initial moment, due to deflection x_0 and the potential energy at the moment under consideration. In Fig. 84 this decrease in potential energy is shown by the shaded area. From eq. (c) we have*

$$dt = \frac{dx}{-\sqrt{2p^2 \int_x^{x_0} f(x)dx}}. \qquad (d)$$

By integration of this equation, the time t as a function of the displacement is obtained,

$$t = \int \frac{dx}{-\sqrt{2p^2 \int_x^{x_0} f(x)dx}}. \qquad (e)$$

Take, for instance, as an example, the case of simple harmonic vibration. Then

$$f(x) = x.$$

From eq. (e), we obtain

$$t = \int \frac{dx}{-p\sqrt{x_0^2 - x^2}} = \int \frac{d\left(\frac{x}{x_0}\right)}{-p\sqrt{1 - \left(\frac{x}{x_0}\right)^2}}$$

or

$$t = \frac{1}{p} \operatorname{arc} \cos \frac{x}{x_0},$$

from which,

$$x = x_0 \cos pt.$$

This result coincides with what we had before for simple harmonic motion.

As a second example, assume,

$$f(x) = x^{2n-1}.$$

Substituting this in eq. (e), we obtain

$$t = \frac{\sqrt{n}}{p} \int \frac{dx}{-\sqrt{x_0^{2n} - x^{2n}}}.$$

* The minus sign is taken because in our case with increase in time x decreases.

The period of vibration will be

$$\tau = \frac{4\sqrt{n}}{p}\frac{1}{x_0{}^{n-1}}\int_0^{x_0}\frac{d\left(\frac{x}{x_0}\right)}{\sqrt{1-\left(\frac{x}{x_0}\right)^{2n}}}. \tag{52}$$

The magnitude of the integral in this equation depends on the value of n and it can be concluded from eq. (52) that only for $n = 1$, i.e., for simple harmonic motion the period does not depend on the initial displacement x_0. For $n = 2$, we have

$$\int_0^{x_0}\frac{d\left(\frac{x}{x_0}\right)}{\sqrt{1-\left(\frac{x}{x_0}\right)^4}} = \int_0^1\frac{du}{\sqrt{1-u^4}} = 1.31.$$

Substituting in eq. (52)

$$\tau = 5.24\frac{\sqrt{2}}{p}\cdot\frac{1}{x_0},$$

i.e., the period of vibration is inversely proportional to the amplitude. Such vibrations we have, for instance, in the case represented in Fig. 78, if the initial tension S in the wire be equal to zero.

In a more general case when

$$f(x) = ax + bx^2 + cx^3$$

a solution of eq. 51 can be obtained by using elliptic functions.* But these solutions are complicated and not suitable for technical applications. Therefore now some graphical and numerical methods for solving eq. (51) will be discussed.

23. Graphical Solution.—In the solution of the general equation (51) two integrations, shown in eq. (*b*) and (*e*) of the previous article must be performed. It is only in the simplest cases that an exact integration of

* Some examples of this kind are discussed in the book "Erzwungene Schwingungen bei veränderlicher Eigenfrequenz," by G. Düffing, Braunschweig, 1918. A general solution of this problem by the use of elliptic functions was given by K. Weierstrass, Monatsberichte der Berliner Akademie, 1866. See also, Gesammelte Werke, Vol. 2, 1895. The application of Bessel's functions in solving the same problem is given in the book by M. J. Akimoff, "Sur les Functions de Bessel a plusieurs variables et leurs applications en mecanique," S. Petersburg, 1929. An approximate solution by using Simpson's formula was discussed by K. Klotter. See Ingenieur-Archiv., Vol. 7, p. 87, 1936.

these is possible, but an approximate graphical solution can always be obtained on the basis of which the period of free vibration for any amplitude can be calculated with a sufficient accuracy.

Let the curve om (Fig. 85) represent to a certain scale the restoring force as a function of the displacement x of the system from its middle position. From eq. (b) (p. 119) it is seen that by plotting the integral curve to the curve om the magnitude of $\dot{x}^2$ as a function of the displacement of x will be obtained. This graphical integration can be performed as follows: The continuous curve om is replaced by a step curve $abdfhlno$ in such a manner as to make $\Delta\, abc = \Delta\, cde$, $\Delta\, efg = \Delta ghk$ and $\Delta\, klm = \Delta\, mno$ so that the area included between the $abdfhln$ line and the x axis becomes equal to that between the om curve and the x axis.

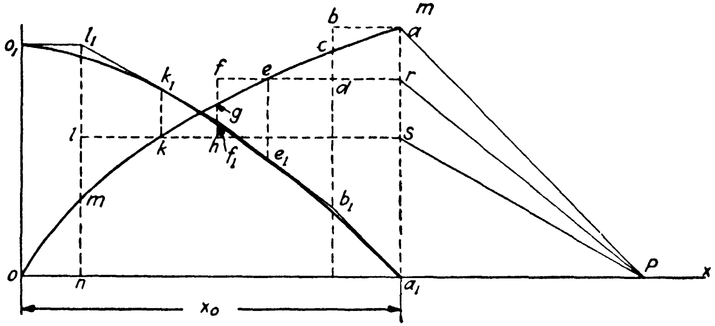

FIG. 85.

A pole distance Pa_1 is now chosen such that it represents unity on the same scale as the ordinates of the om curve and the rays Pa, Pr, Ps are drawn. Making now $a_1b_1 \parallel Pa$, $b_1f_1 \parallel Pr$, $f_1l_1 \parallel Ps$ and $l_1o_1 \parallel Pa_1$, the polygon $a_1b_1f_1l_1o_1$ will be obtained, the slopes of whose sides are equal to the corresponding values of the function represented by $abdfhln$. This means that the $a_1b_1f_1l_1o_1$ line is the integral curve for the $abdfhln$ line. Due to the equality of triangles (see Fig. 85) mentioned above, the sides of the polygon $a_1b_1f_1l_1o_1$ must be tangent to the integral curve of om; the points of tangency being at a_1, e_1, k_1 and o_1. Therefore the curve $a_1e_1k_1o_1$ tangent to the polygon $a_1b_1f_1l_1o_1$ at a_1, e_1, k_1 and o_1 represents the integral curve for the curve om and gives to a certain scale the variation of the kinetic energy of the system during the motion from the extreme position ($x = x_0$) to the middle position ($x = 0$). If the ordinates of the curve om are equal to a certain scale, to $2p^2f(x)$ (see eq. (b), p. 119) and the pole distance Pa_1 is equal to unity to the same scale then the ordinates of the $a_1e_1k_1o_1$ curve, if measured to the same scale as the displacement x_0, give the magnitudes of $\dot{x}^2$. From this the velocity $\dot{x}$ and the inverse quantity $1/\dot{x}$ can readily be calculated and the curve pn representing $1/\dot{x}$ as a function of x can be plotted (see Fig. 86). The time which will be taken by the system to reach its middle position ($x = 0$) from its extreme position ($x = x_0$) will be represented by the following integral (see eq. (e), p. 120)

$$t = \int_{x_0}^{0} -\frac{dx}{\dot{x}}.$$

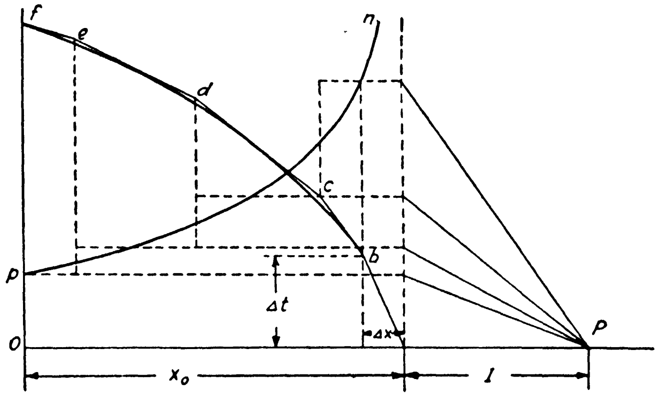

FIG. 86.

This means that t can be obtained by plotting the integral curve of the curve pn (see Fig. 86) exactly in the same manner as explained above. The final ordinate Of, measured to the same scale as x_0, gives the time t. In the case of a system symmetrical about its middle position the time t will represent a quarter of the period of free vibration for the amplitude x_0. It must be noted that for $x = x_0$, $\dot{x} = 0$, i.e., $1/\dot{x}$ becomes infinitely large at this point. In order to remove this difficulty the plotting of the integral curve can be commenced from a certain point b, the small coordinates Δx and Δt of which will be determined on the assumption that at the beginning along a small distance Δx the system moves with a constant acceleration equal to $p^2f(x)$, (see eq. (51), p. 119). Then

$$\Delta x = \frac{\Delta t^2}{2} p^2 f(x_0)$$

and

$$\Delta t = \sqrt{\frac{2\Delta x}{p^2 f(x_0)}}.$$

Another graphical method, developed by Lord Kelvin,* also can be used in discussing the differential equation of non-harmonic vibration. For the general case the differential equation of motion can be presented in the following form

$$\ddot{x} = f(x, t, \dot{x}). \tag{53}$$

The solution of this equation will represent the displacement x as a function of the time t. This function can be represented graphically by time-displacement curve (Fig. 87). In order to obtain a definite solution the initial conditions, i.e., the initial displacement and initial velocity of the system must be known.

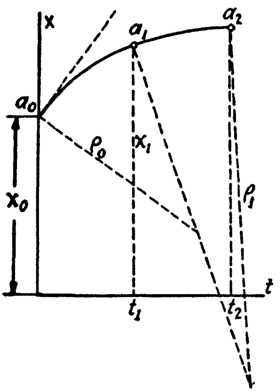

FIG. 87.

Let $x = x_0$ and $\dot{x} = \dot{x}_0$ for $t = 0$.

Then the initial ordinate and initial slope of the time-displacement curve are known. Substituting the initial values of x and $\dot{x}$ in eq. (53), the initial value of $\ddot{x}$ can be calculated. Now from the known equation,

$$\rho = \frac{\sqrt{(1 + \dot{x}^2)^3}}{\ddot{x}}, \tag{a}$$

the radius of curvature ρ_0 at the beginning of the time-displacement curve can be found. By using this radius a small element a_0a_1 of the time-displacement curve can be traced as an arc of a circle (Fig. 87) and the values of the ordinate $x = x_1$ and of the slope $\dot{x} = \dot{x}_1$ at the new point a_1 can be taken from the drawing and the corresponding value of $\ddot{x}$ calculated from eq. (53). Now from eq. (a) the magnitude of $\rho = \rho_1$ will be

* See, Lord Kelvin, On Graphic Solution of Dynamical Problems, Phil. Mag., Vol. 34 (1892). The description of this and several other graphical methods of integrating differential equations can be found in the book "Die Differentialgleichungen des Ingenieurs," by W. Hort (2d ed., 1925), Berlin, which contains applications of these methods to the solution of technical problems. See also H. von Sanden, Practical Mathematical Analysis, New York, 1926. Further development of graphical methods of integration of differential equations with applications to the solution of vibration problems is due to Dr. E. Meissner. See his papers, "Graphische Analysis vermittelst des Linienbildes einer Function," Kommissions verlag Rascher & Co., Zürich, 1932; Schweizerische Bauzeitung, Vol. 104, 1934; Zeitschr. f. angew. Math. u. Mech. Vol. 15, 1935, p. 62.

obtained by the use of which the next element a_1a_2 of the curve can be traced. Continuing this construction, as described, the time-displacement curve will be graphically obtained. The calculations involved can be somewhat simplified by using the angle of inclination of a tangent to the time-displacement curve. Let θ denote this angle, then

$$\dot{x} = \tan\theta \quad \text{and} \quad \ddot{x} = f(x, t, \tan\theta).$$

Substituting in eq. (a)

$$\rho = \frac{\sqrt{(1 + \tan^2\theta)^3}}{f(x, t, \tan\theta)} = \frac{1}{\cos^3\theta f(x, t, \tan\theta)}. \tag{b}$$

In this calculation the square root is taken with the positive sign so that the sign of ρ is the same as the sign of $\ddot{x}$. If $\ddot{x}$ is negative the center of curvature must be taken in such a manner as to obtain the curve convex up (see Fig. 87).

In the case of free vibration and by neglecting damping, eq. (53) assumes the form given in (51) and the graphical integration described above becomes very simple, because the function f depends in this case only on the magnitude of displacement x. Taking for the initial conditions $x = x_0$ and $\dot{x} = 0$ for $t = 0$, the time-displacement curve will have the general form shown in Fig. 88. In the case of a system symmetrical about the middle position the intersection of this curve with the t axis will determine the period τ of the free vibration of the system. The magnitude of τ can always be determined in this manner with an accuracy sufficient for practical applications. In Fig. 88 for instance, the case of a simple harmonic vibration was taken for which the differential equation is

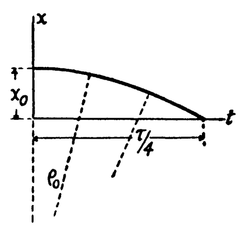

FIG. 88.

$$\ddot{x} + p^2x = 0$$

and the exact solution gives

$$\tau = \frac{2\pi}{p}.$$

Equation (b) for this case becomes

$$\rho = \frac{1}{\cos^3\theta\, p^2x}. \tag{c}$$

The initial displacement x_0 in Fig. 88 is taken equal to 20 units of length and ρ_0 equal to 100 units of length. Then from eq. (*c*) for $\theta = 0$, we obtain

$$\frac{1}{p} = \sqrt{20 \cdot 100} = 44.7 \text{ units.} \qquad (d)$$

The quantity $1/p$ has the dimension of time and the length given by eq. (*d*) should be used in determining the period from Fig. 88. By measuring $\frac{\tau}{4}$ to the scale used for x_0 and ρ, we obtain from this figure

$$\frac{\tau}{4} = 69.5 \text{ units}$$

or by using (*d*)

$$\tau = 1/p \cdot \frac{4 \times 69.5}{44.7} = \frac{6.22}{p}.$$

In this graphical solution only 7 intervals have been taken in drawing the quarter of the period of the time-displacement curve and the result obtained is accurate within 1%.

24. Numerical Solution.—Nonharmonic vibrations as given by equations (51) and (53) can also be solved in a numerical way. Consider as an example free vibration without damping. The corresponding differential equation is

$$\ddot{x} + p^2 f(x) = 0. \qquad (a)$$

Let the initial conditions be

$$x = x_0; \qquad \dot{x} = 0, \qquad \text{for} \qquad t = 0. \qquad (b)$$

By substituting x_0 for x in eq. (*a*) the magnitude of $\ddot{x}_0$ can be calculated. By using the value $\ddot{x}_0$ of the acceleration at $t = 0$ the magnitude of $\dot{x}_1$ and x_1, i.e., the velocity and displacement at any moment t_1 chosen very close to the time $t = 0$ can be calculated. Let Δt denote the small interval of time between the instant $t = 0$ and the instant $t = t_1$. The approximate value of $\dot{x}_1$ and x_1 will then be obtained from the following equations,

$$\dot{x}_1 = \dot{x}_0 + \ddot{x}_0 \Delta t; \quad x_1 = x_0 + \frac{\dot{x}_0 + \dot{x}_1}{2} \Delta t. \qquad (c)$$

Substituting the value x_1 for x in eq. (*a*), the value of $\ddot{x}_1$ will be obtained.

By using this latter value better approximations for $\dot{x}_1$ and x_1 can be calculated from the following equations,

$$\dot{x}_1 = \dot{x}_0 + \frac{\ddot{x}_0 + \ddot{x}_1}{2} \Delta t \quad \text{and} \quad x_1 = x_0 + \frac{\dot{x}_0 + \dot{x}_1}{2} \Delta t. \tag{d}$$

A still better approximation for $\ddot{x}_1$ will now be obtained by substituting the second approximation of x_1 (eq. (*d*) in eq. (*a*). Now, taking the second step, by using x_1, $\dot{x}_1$ and $\ddot{x}_1$ the magnitude of x_2, $\dot{x}_2$, $\ddot{x}_2$ for the time $t = t_2 = 2\Delta t$ can be calculated exactly in the same manner as explained above. By taking the intervals Δt small enough and making the calculations for every value of t twice as explained above in order to obtain the second approximation, this method of numerical integration can always be made sufficiently accurate for practical applications.

In order to show this procedure of calculation and to give some idea of the accuracy of the method we will consider the case of simple harmonic vibration, for which the equation of motion is:

$$\ddot{x} + p^2 x = 0.$$

The exact solution of this equation for the initial conditions (*b*) is

$$x = x_0 \cos pt; \quad \dot{x} = - x_0 p \sin pt. \tag{e}$$

The results of the numerical integration are given in the table below. The length of the time intervals was taken equal to $\Delta t = 1/4p$. Remembering that the period of vibration in this case is $\tau = 2\pi/p$ it is seen that Δt, the interval chosen, is equal approximately to 1/6 of a quarter of the period τ. The second line of the table expresses the initial conditions. Now, for obtaining first approximations for $\dot{x}_1$ and x_1, at the time $t = \Delta t = 1/4p$, equations (*c*) were used. The results obtained are given in the third line of the table. For getting better approximations for $\dot{x}_1$ and x_1, equations (*d*) were used and the results are put in the fourth line of the table. Proceeding in this manner the complete table was calculated. In the last two columns the corresponding values of sin *pt* and cos *pt* proportional to the exact solutions (*e*) are given, so that the accuracy of the numerical integration can be seen directly from the table. We see that the velocities obtained by calculation have always a high accuracy. The largest error in the displacement is seen from the last line of the table and amounts to about 1% of the initial displacement x_0.

These results were obtained by taking only 6 intervals in a quarter of a period. By increasing the number of intervals the accuracy can be increased, but at the same time the number of necessary calculations becomes larger.

TABLE I

NUMERICAL INTEGRATION

t	x	$\dot{x}$	$\ddot{x}$	$\cos pt$	$\sin pt$
0	x_0	0	$-p^2x_0$	1	0
Δt	$.9687\ x_0$	$-\ .2500\ px_0$	$-.9687\ p^2x_0$		
Δt	.9692 “	− .2461 “	−.9692 “	.9689	.2474
$2\Delta t$	.8774 “	− .4884 “	−.8774 “		
$2\Delta t$	.8788 “	− .4769 “	− 8788 “	.8776	.4794
$3\Delta t$	.7321 “	− .6966 “	−.7321 “		
$3\Delta t$	.7344 “	− .6783 “	− 7344 “	.7317	.6816
$4\Delta t$	.5419 “	− .8619 “	−.5419 “		
$4\Delta t$	.5449 “	− .8378 “	−.5449 “	.5403	.8415
$5\Delta t$	.3184 “	− .9740 “	− 3184 “		
$5\Delta t$	.3220 “	− .9457 “	− 3220 “	.3153	.9490
$6\Delta t$	.0755 “	−1.0262 “	−.0755 “		
$6\Delta t$	.0794 “	− .9954 “	−.0794 “	.0707	.9975
$7\Delta t$	−.1719 “	−1.0153 “	−.1719 “		
$7\Delta t$	−.1680 “	− .9838 “	−.1680 “	−.1782	.9840

By using the table the period of vibration also can be calculated. It is seen from the first and second columns that for $t = 6\Delta t$ the time-displacement curve has a positive ordinate equal to $.0794x_0$. For $t = 7\Delta t$ the ordinate of the same curve is negative and equal to $.1680x_0$. The point of intersection of the time-displacement curve with the t axis determines the time equal to a quarter of the period of vibration. By using linear interpolation this time will be found from the equation

$$\frac{1}{4}\tau = 6\Delta t + \Delta t\frac{.0794}{.0794 + .1680} = 6.32\Delta t = \frac{6.32}{4p} = \frac{1.58}{p}.$$

The exact value of the quarter of a period of vibration is $\pi/2p \approx 1.57/p$. It is seen that by the calculation indicated the period of vibration is obtained with an error less than 1%. From this example it is easy to see that the numerical method described can be very useful for calculating the period of vibration of systems having a flexibility which varies with the displacement.*

* A discussion of more elaborate methods of numerical integration of differential equations can be found in the previously mentioned books by W. Hort and by H.

25. Method of Successive Approximations Applied to Free Vibrations.—We begin with the problems in which the non-linearity of the equation of motion is due to the non-linear characteristic of the spring. If the deviation of the spring deformation from Hooke's law is comparatively small, the differential equation of the motion can be represented in the following form:

$$\ddot{x} + p^2x + \alpha f(x) = 0 \tag{54}$$

in which α is a small factor and $f(x)$ is a polynomial of x with the lowest power of x not smaller than 2. In the cases when the arrangement of the system is symmetrical with respect to the configuration of static equilibrium, i.e., for $x = 0$, the numerical value of $f(x)$ must remain unchanged when x is replaced by $-x$, in such cases $f(x)$ must contain odd powers of x only. The simplest equation of this kind is obtained by keeping only the first term in the expression for $f(x)$. Then the equation of motion becomes:

$$\ddot{x} + p^2x + \alpha x^3 = 0. \tag{55}$$

A system of this kind is shown in Fig. 78. Since there are important problems in astronomy which require studies of eqs. (54) and (55), several methods of handling them have been developed.† In the following a general method is discussed for obtaining periodical solutions of eq. (55) by calculating successive approximations.

We begin with the calculation of the second approximation of the solution of eq. (55).‡ Since α is small it is logical to assume, as a first approximation, for x a simple harmonic motion with a circular frequency p_1, which differs only little from the frequency p. We then put

$$p^2 = p_1^2 + (p^2 - p_1^2), \tag{a}$$

where $p^2 - p_1^2$ is a small quantity. Substituting (a) in eq. (55) we obtain:

$$\ddot{x} + p_1^2x + (p^2 - p_1^2)\,x + \alpha x^3 = 0. \tag{b}$$

Assuming that at the initial instant, $t = 0$, we have $x = a$, $\dot{x} = 0$, the harmonic motion satisfying these initial conditions is given by

$$x = a \cos p_1t. \tag{c}$$

---

von Sanden (p. 124). See also books by Runge-König, "Vorlesungen über numerisches Rechnen," Berlin, 1924, and A. N. Kriloff, Approximate Numerical Integration of Ordinary Differential Equations, Berlin, 1923 (Russian).

† These methods are discussed in the paper by A. N. Kriloff, Bulletin of the Russian Academy of Sciences, 1933, No. 1, p. 1. The method which is described in the following discussion is developed principally by A. Lindsted, Mémoires de l' Acad. des Sciences de St. Petersbourg, VII serie, Vol. 31, 1883, and by A. M. Liapounoff in his doctor thesis dealing with the general problem on stability of motion, Charkow, 1892 (Russian).

‡ Such an approximation was obtained first by M. V. Ostrogradsky, see Mémoires de l'Acad. des Sciences de St. Petersbourg, VI serie, Vol. 3, 1840. A similar solution was given also by Lord Rayleigh in his Theory of Sound, Vol. 1, 1894, p. 77. The incompleteness of both these solutions is discussed in the above mentioned paper by A. N. Kriloff.

This represents the first approximation to the solution of the eq. (55) for the given initial conditions.

Substituting this expression for x into the last two terms of eq. (b), which are small, we obtain:

$$\ddot{x} + p_1^2 x = -a(p^2 - p_1^2)\cos p_1 t - \alpha a^3 \cos^3 p_1 t$$

or, by using the relation

$$4\cos^3 p_1 t = \cos 3p_1 t + 3\cos p_1 t$$

we find

$$\ddot{x} + p_1^2 x = -\left[a(p^2 - p_1^2) + \frac{3\alpha a^3}{4}\right]\cos p_1 t - \frac{\alpha a^3}{4}\cos 3p_1 t. \qquad (d)$$

Thus we obtain apparently an equation of forced vibration for the case of harmonic motion without damping. The first term on the right side of the equation represents a disturbing element which has the same frequency as the frequency of the natural vibrations of the system. To eliminate the possibility of resonance we employ an artifice that consists in choosing a value of p_1 that will make:*

$$a(p^2 - p_1^2) + \frac{3\alpha a^3}{4} = 0.$$

From this equation we obtain:

$$p_1^2 = p^2 + \frac{3\alpha a^2}{4}. \qquad (e)$$

Combining eqs. (d) and (e) we find the following general solution for x

$$x = C_1 \cos p_1 t + C_2 \sin p_1 t + \frac{\alpha a^3}{32 p_1^2}\cos 3p_1 t.$$

To satisfy the assumed initial conditions we must put

$$C_1 = a - \frac{\alpha a^3}{32 p_1^2}$$

and

$$C_2 = 0$$

in this solution. From this it follows that the second approximation for x is

$$x = \left(a - \frac{\alpha a^3}{32 p_1^2}\right)\cos p_1 t + \frac{\alpha a^3}{32 p_1^2}\cos 3p_1 t. \qquad (56)$$

It is seen that due to presence in eq. (55) of the term involving x^3 the solution is no longer a simple harmonic motion proportional to $\cos p_1 t$. A higher harmonic, proportional to $\cos 3p_1 t$ appears, so that the actual time-displacement curve is not a cosine curve. The magnitude of the deviation from the simple harmonic curve depends on the magnitude of the factor α. Moreover, the fundamental frequency of the vibration,

* This manner of calculation p_1 represents an essential feature of the method of successive approximation. If the factor before $\cos p_1 t$ in eq. (d) is not eliminated a term in the expresssion for x will be obtained which increases indefinitely with the time t.

as we see from eq. (e), is no longer constant. It depends on the amplitude of vibrations a, and it increases with the amplitude in the case when α is positive. Such conditions prevail in the case represented by Fig. 78.

Expressions (e) and (56) can be put into the following forms

$$
\begin{aligned}
p^2 &= p_1^2 + c_1\alpha \\
x &= \varphi_0 + \alpha\varphi_1
\end{aligned}
\tag{f}
$$

where

$$
c_1 = -\frac{3a^2}{4}, \qquad \varphi_0 = a\cos p_1 t
$$

$$
\varphi_1 = \frac{a^3}{32p_1^2}(\cos 3\,p_1t - \cos p_1t).
$$

Thus the approximate expressions (f) for the frequency and for the displacement contain the small quantity α to the first power. If we wish to get further approximations we take, instead of expressions (f), the series:

$$
\begin{aligned}
x &= \varphi_0 + \alpha\varphi_1 + \alpha^2\varphi_2 + \alpha^3\varphi_3 + \cdots \\
p^2 &= p_1^2 + c_1\alpha + c_2\alpha^2 + c_3\alpha^3 + \cdots
\end{aligned}
\tag{g}
$$

which contain higher powers of the small quantity α. In these series $\varphi_0, \varphi_1, \varphi_2, \cdots$ are unknown functions of time t, p_1 is the frequency, which will be determined later, and $c_1, c_2, \cdots$ are constants which will be chosen so as to eliminate condition of resonance as was explained above in the calculation of the second approximation. By increasing the number of terms in expressions (g) we can calculate as many successive approximations as we desire. In the following discussion we limit our calculations by omitting all the terms containing α in a power higher than the third. Substituting expressions (g) into eq. (55) we obtain:

$$
\begin{aligned}
\ddot{\varphi}_0 + \alpha\ddot{\varphi}_1 + \alpha^2\ddot{\varphi}_2 + \alpha^3\ddot{\varphi}_3 + (p_1^2 + c_1\alpha + c_2\alpha^2 + c_3\alpha^3)(\varphi_0 + \alpha\varphi_1 + \alpha^2\varphi_2 + \alpha^3\varphi_3) \\
+ \alpha(\varphi_0 + \alpha\varphi_1 + \alpha^2\varphi_2 + \alpha^3\varphi_3)^3 = 0.
\end{aligned}
\tag{h}
$$

After making the indicated algebraic operations and neglecting all the terms containing α to a power higher than the third, we can represent eq. (h) in the following form:

$$
\begin{aligned}
\ddot{\varphi}_0 + p_1^2\varphi_0 + \alpha(\ddot{\varphi}_1 + p_1^2\varphi_1 + c_1\varphi_0 + \varphi_0^3) + \alpha^2(\ddot{\varphi}_2 + p_1^2\varphi_2 + c_2\varphi_0 + c_1\varphi_1 + 3\varphi_0^2\varphi_1) \\
+ \alpha^3(\ddot{\varphi}_3 + p_1^2\varphi_3 + c_3\varphi_0 + c_2\varphi_1 + c_1\varphi_2 + 3\varphi_0^2\varphi_2 + 3\varphi_0\varphi_1^2) = 0.
\end{aligned}
\tag{i}
$$

This equation must hold for any value of the small quantity α which means that each factor for each of the tree powers of α must be zero. Thus eq. (i) will split in the following system of equations:

$$
\begin{aligned}
&\ddot{\varphi}_0 + p_1^2\varphi_0 = 0, \\
&\ddot{\varphi}_1 + p_1^2\varphi_1 = -c_1\varphi_0 - \varphi_0^3, \\
&\ddot{\varphi}_2 + p_1^2\varphi_2 = -c_2\varphi_0 - c_1\varphi_1 - 3\varphi_0^2\varphi_1, \\
&\ddot{\varphi}_3 + p_1^2\varphi_3 = -c_3\varphi_0 - c_2\varphi_1 - c_1\varphi_2 - 3\varphi_0^2\varphi_2 - 3\varphi_0\varphi_1^2.
\end{aligned}
\tag{j}
$$

Taking the same initial conditions as before, i.e., for $t = 0$,

$$x = a \quad \text{and} \quad \dot{x} = 0$$

and substituting for x from eq. (g), we obtain:

$$\varphi_0(0) + \alpha\varphi_1(0) + \alpha^2\varphi_2(0) + \alpha^3\varphi_3(0) = a$$
$$\dot{\varphi}_0(0) + \alpha\dot{\varphi}_1(0) + \alpha^2\dot{\varphi}_2(0) + \alpha^3\dot{\varphi}_3(0) = 0.$$

Again, since these equations must hold for any magnitude of α, we have:

$$\begin{array}{ll} \varphi_0(0) = a & \dot{\varphi}_0(0) = 0 \\ \varphi_1(0) = 0 & \dot{\varphi}_1(0) = 0 \\ \varphi_2(0) = 0 & \dot{\varphi}_2(0) = 0 \\ \varphi_3(0) = 0 & \dot{\varphi}_3(0) = 0. \end{array} \qquad (k)$$

Considering the first of eqs. (j) and the corresponding initial conditions represented by the first row of the system (k) we find as before

$$\varphi_0 = a \cos p_1 t. \qquad (l)$$

Substituting this first approximation into the right side of the second of eqs. (j) we obtain

$$\ddot{\varphi}_1 + p_1^2\varphi_1 = -c_1 a \cos p_1 t - a^3 \cos^3 p_1 t = -(c_1 a + \tfrac{3}{4}a^3)\cos p_1 t - \tfrac{1}{4}a^3 \cos 3p_1 t.$$

To eliminate the condition of resonance we will choose the constant c_1 so as to make the first term on the right side of the equation equal to zero. Then

$$c_1 a + \tfrac{3}{4}a^3 = 0$$

and we find

$$c_1 = -\tfrac{3}{4}a^2. \qquad (m)$$

The general solution for φ_1 then becomes

$$\varphi_1 = C_1 \cos p_1 t + C_2 \sin p_1 t + \frac{1}{32}\frac{a^3}{p_1^2}\cos 3p_1 t.$$

To satisfy the initial conditions given by the second row of the system (k), we put

$$C_1 + \frac{a^3}{32p_1^2} = 0$$
$$C_2 = 0.$$

Thus

$$\varphi_1 = \frac{a^3}{32p_1^2}(\cos 3p_1 t - \cos p_1 t). \qquad (n)$$

If we limit our calculations to the second approximation and substitute expressions (l), (m) and (n) into expressions (g), we obtain

$$x = a \cos p_1 t + \frac{\alpha a^3}{32p_1^2}(\cos 3\,p_1 t - \cos p_1 t) \qquad (o)$$

where

$$p_1^2 = p^2 + \tfrac{3}{4}a^2\alpha. \qquad (p)$$

These results coincide entirely with expressions (f) which were previously obtained (see p. 131).

To obtain the third approximation we substitute the expressions (l), (m) and (n) into the right side of the third of equations (j) and obtain

$$\ddot{\varphi}_2 + p_1^2\varphi_2 = -c_2 a\cos p_1 t + \tfrac{3}{4}a^2\cdot\frac{a^3}{32p_1^2}(\cos 3p_1 t - \cos p_1 t)$$
$$- 3a^2\cos^2 p_1 t\cdot\frac{a^3}{32p_1^2}(\cos 3p_1 t - \cos p_1 t).$$

By using formulae for trigonometric functions of multiple angles we can write this equation in the following form:

$$\ddot{\varphi}_2 + p_1^2\varphi_2 = -a\left(c_2 + \frac{3}{128}\frac{a^4}{p_1^2}\right)\cos p_1 t - \frac{3}{128}\frac{a^5}{p_1^2}(\cos 3p_1 t + \cos 5p_1 t).$$

Again, to eliminate the condition of resonance, we put

$$c_2 = -\frac{3}{128}\frac{a^4}{p_1^2}. \tag{r}$$

Then the general solution for φ_2 becomes

$$\varphi_2 = C_1\cos p_1 t + C_2\sin p_1 t + \frac{3}{1024}\frac{a^5}{p_1^4}\cos 3p_1 t + \frac{1}{1024}\frac{a^5}{p_1^4}\cos 5p_1 t.$$

By using the third row of the system (k), the constants of integration are

$$C_1 = -\frac{1}{256}\frac{a^5}{p_1^4}.$$
$$C_2 = 0.$$

Thus we obtain

$$\varphi_2 = \frac{1}{1024}\frac{a^5}{p_1^4}(\cos 5p_1 t + 3\cos 3p_1 t - 4\cos p_1 t). \tag{s}$$

If we limit the series (g) to terms containing α and α^2, we obtain the third approximation by using the above results for φ_0, φ_1, φ_2, c_1 and c_2:

$$x = a\cos p_1 t + \frac{\alpha a^3}{32p_1^2}(\cos 3p_1 t - \cos p_1 t) + \frac{\alpha^2 a^5}{1024p_1^4}(\cos 5p_1 t + 3\cos 3p_1 t - 4\cos p_1 t) \tag{t}$$

where p_1 is now determined by the equation

$$p_1^2 = p^2 + \frac{3}{4}a^2\alpha + \frac{3}{128}\frac{a^4\alpha^2}{p_1^2}. \tag{u}$$

Substituting the expressions for φ_0, φ_1, φ_2, c_1 and c_2 in the last of eqs. (j), and proceeding as before, we finally obtain the fourth approximation

$$x = a\cos p_1 t + \frac{\alpha}{32}\frac{a^3}{p_1^2}(\cos 3p_1 t - \cos p_1 t) + \frac{\alpha^2}{1024}\frac{a^5}{p_1^4}(\cos 5p_1 t + 3\cos 3p_1 t - 4\cos p_1 t)$$
$$+ \frac{\alpha^3}{32768}\frac{a^7}{p_1^6}(\cos 7p_1 t + 3\cos 5p_1 t - 3\cos 3p_1 t - \cos p_1 t), \tag{v}$$

in which

$$p_1^2 = p^2 + \frac{3}{4}\alpha a^2 + \frac{3}{128}\alpha^2\frac{a^4}{p_1^2} - \frac{3}{1024}\alpha^3\frac{a^6}{p_1^4}. \tag{w}$$

Since in all our calculations we have omitted terms containing α to a power higher than the third, we simplify eq. (w) by substituting in the third term on the right side the second approximation (p) for p_1 and in the last term of the same side substituting p for p_1. Thus we obtain

$$p_1^2 = p^2 + \frac{3}{4}\alpha a^2 + \frac{3}{128}\alpha^2\frac{a^4}{p^2 + \frac{3}{4}\alpha a^2} - \frac{3}{1024}\alpha^3\frac{a^6}{p^4},$$

from which

$$p_1^2 = p^2 + \frac{3}{4}\alpha a^2 + \frac{3}{128}\alpha^2\frac{a^4}{p^2} - \frac{21}{1024}\alpha^3\frac{a^6}{p^4}.$$

We see that the frequency p_1 depends on the amplitude a of the vibration. The time displacement curve is not a simple cosine curve; it contains, according to expression (v), higher harmonics, the amplitudes of which, for small values of α, are rapidly diminishing as the order of the harmonic increases.

Let us apply the method to the case of vibration of a theoretical pendulum. Equation of motion in this case is Fig. 79 (see p. 116)

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0.$$

Developing $\sin\theta$ in the series and using only the two first terms of this series we obtain

$$\ddot{\theta} + \frac{g}{l}\theta - \frac{g}{6l}\theta^3 = 0.$$

Taking for the frequency the second approximation (e) and denoting by θ_0 the angular amplitude, we find

$$p_1^2 = \frac{g}{l} - \frac{g}{8l}\theta_0^2.$$

Thus the period of oscillation is

$$\tau = \frac{2\pi}{p_1} = 2\pi\sqrt{\frac{l}{g}}\frac{1}{\sqrt{1 - \frac{1}{8}\theta_0^2}} \approx 2\pi\sqrt{\frac{l}{g}}\left(1 + \tfrac{1}{16}\theta_0^2\right).$$

This formula is a very satisfactory one for angles of swing smaller than one radian.

The method of successive approximations, applied to solutions of eq. (55), can be used also in the more general case of eq. (54).

The same method can be employed also in studying non-harmonic vibrations in which the non-linearity of the equation of motion is due to a non-linear expression for the damping force. As an example let us consider the case when the damping force is proportional to the square of the velocity. The equation of motion is then:

$$\ddot{x} + p^2x \mp \alpha\dot{x}^2 = 0.$$

The minus sign must be taken when the velocity is in the direction of the negative x axis and the plus sign for the velocity in the direction of the positive x axis. Taking $x = a$ and $\dot{x} = 0$ at the initial instant ($t = 0$), we have for the first half of the oscillation the equation

$$\ddot{x} + p^2x - \alpha\dot{x}^2 = 0. \tag{a'}$$

Limiting our calculations to terms containing α^2, we put, as before,

$$x = \varphi_0 + \alpha\varphi_1 + \alpha^2\varphi_2 \tag{b'}$$

$$p^2 = p_1^2 + c_1\alpha + c_2\alpha^2.$$

Substituting in eq. $(a)'$ and neglecting all terms containing α to powers higher than the second, we obtain the equation

$$\ddot{\varphi}_0 + p_1^2\varphi_0 + \alpha(\ddot{\varphi}_1 + p_1^2\varphi_1 - \dot{\varphi}_0^2) + \alpha^2(\ddot{\varphi}_2 + p_1^2\varphi_2 + c_1\varphi_1 + c_2\varphi_0 - 2\dot{\varphi}_0\dot{\varphi}_1) = 0$$

from which it follows that:

$$\begin{aligned} \ddot{\varphi}_0 + p_1^2\varphi_0 &= 0 \\ \ddot{\varphi}_1 + p_1^2\varphi_1 &= \dot{\varphi}_0^2 \\ \ddot{\varphi}_2 + p_1^2\varphi_2 &= -c_1\varphi_1 - c_2\varphi_0 + 2\dot{\varphi}_0\dot{\varphi}_1. \end{aligned} \tag{c'}$$

The initial conditions give

$$\begin{aligned} \varphi_0(0) &= a & \dot{\varphi}(0) &= 0 \\ \varphi_1(0) &= 0 & \dot{\varphi}_1(0) &= 0 \\ \varphi_2(0) &= 0 & \dot{\varphi}_2(0) &= 0. \end{aligned} \tag{d'}$$

From the first of equations $(c)'$ and by using the first row of conditions $(d)'$, we obtain the first approximation

$$\varphi_0 = a\cos p_1t.$$

Substituting this into the right side of the second of equations $(c)'$, we obtain:

$$\ddot{\varphi}_1 + p_1^2\varphi_1 = a^2p_1^2\sin^2 pt = \tfrac{1}{2}a^2p_1^2(1 - \cos 2p_1t).$$

The solution of this equation, satisfying the initial conditions is then:

$$\varphi_1 = \tfrac{1}{2}a^2 - \tfrac{2}{3}a^2\cos p_1t + \tfrac{1}{6}a^2\cos 2p_1t.$$

Substituting φ_0 and φ_1 in the right side of the third of eqs. $(c)'$ we obtain

$$\begin{aligned} \ddot{\varphi}_2 + p_1^2\varphi_2 = -c_2a\cos p_1t &- c_1(\tfrac{1}{2}a^2 - \tfrac{2}{3}a^2\cos p_1t + \tfrac{1}{6}a^2\cos 2p_1t) \\ &- 2a^3p_1^2\sin p_1t(\tfrac{2}{3}\sin p_1t - \tfrac{1}{3}\sin 2p_1t). \end{aligned} \tag{e'}$$

We have on the right side of this equation two constants c_1 and c_2 and since there will be only one condition for the elimination of the possibility resonance, one of these constants can be taken arbitrarily. The simplest assumption is that $c_1 = 0$. Then eq. $(e)'$ can be represented in the following form:

$$\begin{aligned} \ddot{\varphi}_2 + p_1^2\varphi_2 = (-c_2a + \tfrac{1}{3}p_1^2a^3)\cos p_1t &- \tfrac{2}{3}a^3p_1^2 \\ &+ \tfrac{2}{3}a^3p_1^2\cos 2p_1t - \tfrac{1}{3}a^3p_1^2\cos 3p_1t. \end{aligned} \tag{f'}$$

To eliminate the resonance condition we put

$$-c_2a + \tfrac{1}{3}p_1^2a^3 = 0$$

or

$$c_2 = \tfrac{1}{3}p_1^2a^2. \tag{g'}$$

Then the general solution of eq. $(f)'$ is

$$\varphi_2 = C_1 \cos p_1 t + C_2 \sin p_1 t - \tfrac{2}{3} a^3 - \tfrac{2}{9} a^3 \cos 2p_1 t + \tfrac{1}{24} a^3 \cos 3p_1 t.$$

To satisfy the initial conditions, represented by the third row of the system $(d)'$ we must put

$$C_1 = \tfrac{61}{72} a^3, \qquad C_2 = 0,$$

and finally we obtain

$$\varphi_2 = -\tfrac{2}{3} a^3 + \frac{a^3}{72}(61 \cos p_1 t - 16 \cos 2p_1 t + 3 \cos 3p_1 t).$$

Substituting φ_0, φ_1, φ_2, c_1 and c_2 in expressions $(b)'$ we obtain

$$x = a \cos p_1 t + \frac{\alpha a^2}{6}(3 - 4 \cos p_1 t + \cos 2p_1 t) - \frac{\alpha^2 a^3}{72}(48 - 61 \cos p_1 t + 16 \cos 2p_1 t - 3 \cos 3p_1 t) \qquad (h)'$$

and

$$p^2 = p_1^2 + c_1\alpha + c_2\alpha^2 = p_1^2 + \tfrac{1}{3} p_1^2 a^2 \alpha^2$$

from which

$$p_1 = \frac{p}{\sqrt{1 + \tfrac{1}{3} a^2\alpha^2}}. \qquad (i)'$$

The time required for half a cycle is

$$\frac{\tau_1}{2} = \frac{\pi}{p_1} = \frac{\pi}{p}\sqrt{1 + \tfrac{1}{3} a^2\alpha^2} \approx \frac{\pi}{p}(1 + \tfrac{1}{6} a^2\alpha^2) \qquad (j)'$$

and the displacement of the system at the end of the half cycle is obtained from expression $(h)'$ by substituting $p_1 t = \pi$ into it. Then

$$(x)_{t=\pi/p_1} = a_1 = -a + \tfrac{4}{3}\alpha a^2 - \tfrac{16}{9}\alpha^2 a^3. \qquad (k)'$$

Beginning now with the initial conditions $x = a_1$, $\dot{x} = 0$ and using formulae $(j)'$ and $(k)'$, we will find that the time required for the second half of the cycle is

$$\frac{\tau_2}{2} \approx \frac{\pi}{p}(1 + \tfrac{1}{6} a_1^2\alpha^2)$$

and the displacement of the system at the end of the cycle is

$$a_2 = -a_1 + \tfrac{4}{3}\alpha a_1^2 - \tfrac{16}{9}\alpha^2 a_1^3.$$

Thus we obtain oscillations with gradually decreasing amplitudes.*

* Another method of solving the problem on vibrations with damping proportional to the square of velocity is given by Burkhard, Zeitschr. f. Math. u. Phys., Vol. 68, p. 303, 1915. Tables for handling vibration problems with non-linear damping containing a term proportional to the square of velocity have been calculated by W. E. Milne in Univ. of Oregon Publications, Mathematical Series, Vol. 1, No. 1, and Vol. 2, No. 2.

26. Forced Non-Linear Vibrations.—Neglecting damping and assuming that the spring of a vibrating system has a non-linear characteristic, we may represent the differential equation of motion for forced vibrations in the following form:

$$\ddot{x} + p^2x + \alpha f(x) = F(t) \tag{a}$$

in which $F(t)$ is the disturbing force per unit mass of the vibrating body and $f(x)$ is a polynomial determined by the spring characteristic. We assume that the vibrating system is symmetrical with respect to the position of equilibrium, i.e., $f(x)$ contains only terms with odd powers of x. Limiting our discussion to the case when $f(x) = x^3$ and assuming that the disturbing force is proportional to $\cos \omega t$, eq. (a) reduces to the following:

$$\ddot{x} + p^2x + \alpha x^3 = q \cos \omega t. \tag{b}$$

This is a non-linear equation, the general solution of which is unknown. In our investigation we will use approximate methods. From the non-linearity of the equation we conclude that the method of superposition of vibrations which was always applicable in problems discussed in the first chapter does not longer hold, and that if the free vibrations of the system as well as its forced vibrations can be found, the sum of these two motions does not give the resultant vibration. Again, if there are several disturbing forces the resultant forced vibration cannot be obtained by summing up vibrations produced by each individual force as it does in the case of a spring with linear characteristics (see Art. 18).

To simplify the problem we will discuss here only the steady forced vibrations and we will disregard the free vibrations that depend on the initial conditions. We will assume also that α is small, i.e., that the spring approximately follows Hooke's law in the case of small amplitudes. Regarding the vibrations we assume that under the action of a disturbing force, $q \cos \omega t$, a steady forced vibration of the same frequency as the disturbing force will be established, moreover that the motion will be in phase with the disturbing force or with a phase difference equal to π. Let this forced vibration be

$$x = a \cos \omega t. \tag{c}$$

To determine the amplitude a of this vibration we use eq. (b) and take for a such a magnitude as to satisfy this equation when the vibrating system is in an extreme position, i.e. when $\cos \omega t = \pm 1$. Substituting (c) into eq. (b) we obtain in this way the following equation for determining a.

$$p^2a + \alpha a^3 = q + a\omega^2. \tag{d}$$

The left side of the equation represents the force exerted by the spring for an extreme position of the vibrating system, and the right side is the sum of the disturbing force and the inertia force for the same position. All these forces are taken per unit mass of the vibrating body. Proceeding in this way we satisfy eq. (b) for the instants when the system is in extreme positions. The equation will be satisfied also when the system is passing through the middle position since for such a position $\cos \omega t = 0$ and all terms of eq. (b) vanish. For other positions of the vibrating system eq. (b) usually will not be satisfied and the actual motion will not be the simple harmonic motion represented by eq. (c). To find an approximate expression for the actual motion we substitute expression (c) for x in eq. (b). Thus we obtain

$$\ddot{x} = q \cos \omega t - p^2 a \cos \omega t - \alpha a^3 \cos^3 \omega t$$

or by using the formula

$$\cos^3 \omega t = \tfrac{1}{4} (\cos 3\omega t + 3 \cos \omega t)$$

we find

$$\ddot{x} = \left(q - p^2 a - \frac{3\alpha a^3}{4}\right) \cos \omega t - \frac{\alpha a^3}{4} \cos 3\omega t.$$

Integrating this equation we have

$$x = \frac{1}{\omega^2} (-q + p^2 a + \tfrac{3}{4} \alpha a^3) \cos \omega t + \frac{\alpha a^3}{36\omega^2} \cos 3\omega t. \tag{e}$$

It is seen that the vibration is no longer a simple harmonic motion. It contains a term proportional to $\cos 3\omega t$ representing a higher harmonic. The amplitude of this vibration is

$$x_{\max} = \frac{1}{\omega^2} (-q + p^2 a + \tfrac{3}{4} \alpha a^3) + \frac{\alpha a^3}{36\omega^2}. \tag{f}$$

For small values of α this amplitude differs only by a small quantity from the value a as obtained from eq. (d). Sometimes eq. (f) is used for determining the maximum amplitude.* Then, by neglecting the last term on the right side of this equation, we obtain

$$a = \frac{1}{\omega^2} (-q + p^2 a + \tfrac{3}{4} \alpha a^3), \tag{g}$$

* See the book by G. Duffing, "Erzwungene Schwingungen bei veränderlicher Eigenfrequenz," p. 40, Braunschweig, 1918. The justification of such an assumption will be seen from the discussion of successive approximations to the solution of eq. (b), see p. 147.

which differs from eq. (d) only in the small term containing α as a factor.

For determining the amplitude a of forced vibrations a graphical solution of eq. (d) can be used. Taking amplitudes a as abscissas and forces per unit mass as ordinates, the left side of eq. (d) will be represented by curves $OA_1A_2A_3$ and OB_3BC_3, which give the spring's characteristic, Fig. 89. The right side of the same equation can be represented by a

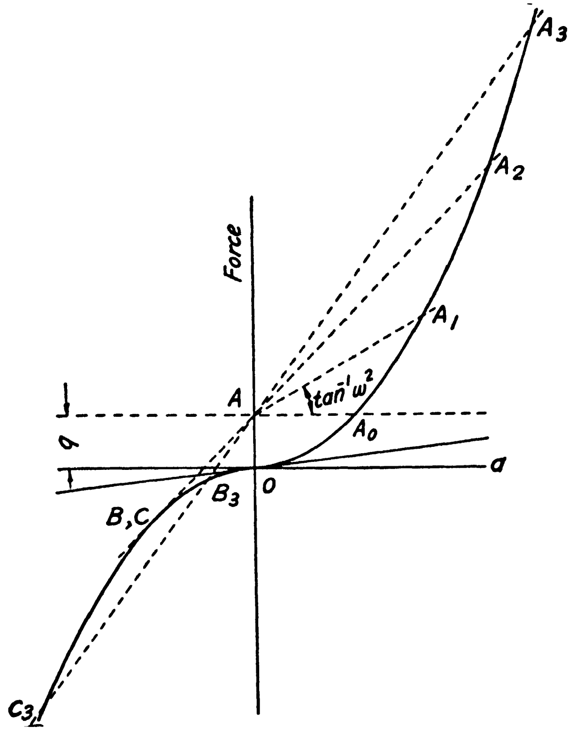

FIG. 89.

straight line with a slope ω^2 and intersecting the ordinate axis at a point A such that OA represents the magnitude q of the disturbing force per unit mass. The straight lines AA_1, AA_2, and AA_3 in the figure are such lines constructed for three different values of the frequency ω. The abscissas of the intersection points A_1, A_2, A_3 give the solutions of the equation (d) and represent the amplitudes of forced vibrations for various frequencies of the disturbing force. It is seen that for smaller values of ω there is only one intersection point, such as point A_1 in the figure, and we obtain only one value for the amplitude of the forced vibration. For the value of ω corresponding to the line AA_2 we have intersection point at A_2 and a point of tangency at B. For higher values of ω we find three points of intersection such as points A_3, B_3 and C_3 in the figure. Thus there are three different values of the amplitude a satisfying eq. (*d*).

Before we go into a discussion of the physical significance of these different solutions, let us introduce another way of graphical representation

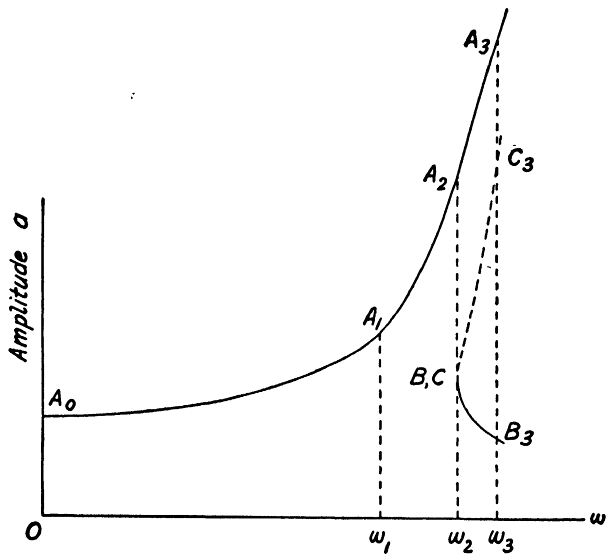

FIG. 90.

of the relation between the amplitude of forced vibrations and the frequency of the disturbing force. We take frequencies ω as abscissas and the corresponding amplitudes a, obtained from Fig. 89, as ordinates. In such way the curves in Fig. 90 have been drawn. The upper curve $A_0A_1A_2A_3$ corresponds to the intersection points A_0, A_1, A_2, A_3 in Fig. 89 and the lower curve C_3BB_3 corresponds to the intersection points such as C_3, B, B_3 in the same figure. It is seen that the upper curve $A_0A_1A_2A_3$ in Fig. 89 corresponds to positive values of a, and we have vibrations in *phase* with the acting force $q \cos \omega t$. For the lower curve B_1, B_3 the amplitudes a are negative and the motion is therefore π radians out of phase with respect to the acting force. In general the curves in Fig. 90 correspond to the non-linear forced vibrations in the same way that the curves in Fig. 10 correspond to the case of simple harmonic motion. By using these curves the amplitude of forced vibrations for any frequency ω of the disturbing force can be obtained. In the case of simple harmonic motion for each value of ω there is only one value of the amplitude, but in the case of non-linear vibrations the problem is more complicated. For frequencies smaller than ω_2 there is again only one value of the amplitude corresponding to a vibration in phase with the disturbing force as for simple harmonic motion. However, for frequencies larger than ω_2 there are three possible solutions; the one with the largest amplitude, is in phase with the force, while the two others are π radians out of phase with the disturbing force. The experiments show* that when we increase the frequency ω of the disturbing force very slowly we obtain first vibrations in phase with the force as given by the curve $A_0A_1A_2A_3$ in Fig. 90. At a certain value of ω, say ω_3, which is larger than ω_2, the motion changes rather abruptly so that instead of having vibrations of comparatively large amplitude $\overline{A_3\omega_3}$ and in phase with the force, we have a much smaller vibration of an amplitude $\overline{\omega_3B_3}$ and with a phase difference π. Vibrations with amplitudes given by the branch CC_3 of the curve, indicated in the Fig. 90 by the dotted line, do not occur at all in the experiments with non-linear forced vibrations. The theoretical explanation of this may be found in the fact that vibrations represented by curves A_0A_3 and BB_3 are stable vibrations,† thus if an accidental force produces a small disturbance from these vibrations, the system will always have a tendency to come back to its original vibration. Vibration given by the dotted line CC_3 is unstable, which means that if a small deviation from this motion is produced by a slight external disturbance, the tendency of the deviations will be to increase so that finally a vibration corresponding in amplitudes to the branch BB_3 or to the branch A_2A_3 of the curve will be built up.

In our discussion it was always assumed that q, the maximum of the pulsating disturbing force remains constant. By using the construction explained in Fig. 89, the amplitude of a forced vibration for any value of q can be determined and the curves similar to those given in Fig. 90 can be plotted. Several curves of this kind are shown in Fig. 91. If, finally,

* The first experiments of this kind which cleared up the significance of the three different possible solutions, discussed above, were made by working with electric current vibrations by O. Martienssen, Phys. Zeitschr., Vol. 11, p. 448, 1910. The same kind of mechanical vibrations were studied by G. Duffing, *loc. cit.*, p. 138.

† A theoretical discussion of the stability of the above mentioned three different types of vibrations was given by E. V. Appleton in his study of "The Motion of a Vibration Galvanometer," see Phil. Mag., ser. 6, Vol. 47, p. 609, 1924. A general discussion on stability of non-linear systems will be found in paper by E. Trefftz, Math. Ann v. 95, p. 307, 1925.

q is taken equal to zero, we obtain the free vibrations of the non-linear system, discussed in the previous article. The frequencies of the free vibrations for various amplitudes are obtained, as stated before, by drawing inclined lines through the point 0 in Fig. 89 and by determining the abscissas of their points of intersection with the curve $OA_1A_2A_3$. It is seen that there is a limiting value ω_0 of the frequency which is determined by the slope of the tangent at 0 to the curve OA_3, Fig. 89. This limiting value is the frequency of the free vibrations of an infinitely small amplitude. For such vibrations the term αx^3 in eq. (b) can be neglected as an

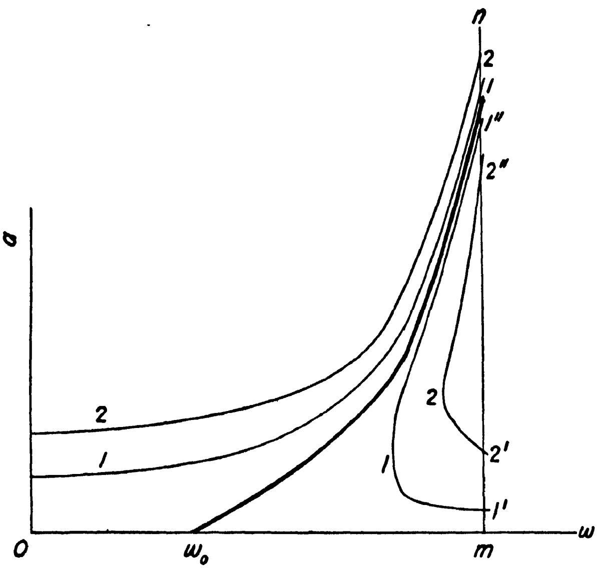

FIG. 91.

infinitely small quantity of a higher order from which we conclude that $\omega_0 = p$. With an increase in amplitudes the frequencies also increase and the relation between a and ω for free vibrations is given in Fig. 91 by the heavy line. From the curves of Fig. 91 some additional information regarding stable and unstable vibrations can be obtained. Focusing our attention upon a constant frequency, corresponding to a vertical line, say mn, that intersects all the curves, and considering the intersection points 1, 2, of this vertical with the stable vibration curves lying above the heavy line, we may conclude that if the maximum of the pulsating force be increased the amplitude of the forced vibration will also increase. The same conclusion can be made regarding the points of intersection 1′, 2′, on the lower portions of the curves below the heavy line which also correspond to the stable conditions of vibrations. However, when we consider points 1″, 2″, on those portions of the curves corresponding to the unstable condition of motion, it is seen from the figure that an increase in the disturbing force produces a decrease in the amplitude of vibration. We know from the previously mentioned experiments that this kind of motion actually does not occur and what really happens is that at certain frequencies the amplitudes given by points 1, 2, are abruptly changed to amplitudes given by points 1′, 2′. The frequencies at which this change of type of motion takes place depend on the amount of damping in the system as well as on the degree of steadiness of the disturbing force.

To simplify our discussion damping was neglected in the derivation of eq. (*b*). If we take damping into consideration and assume that it is proportional to the velocity of motion, we can again determine the amplitude of vibrations by an approximate method similar to the one used above.* Due to damping the curves of Fig. 90 will be rounded as shown in Fig. 92. It is seen that the question of instability arises only in the cases when the frequency of the disturbing force is in the region $\omega_2 < \omega < \omega_3$. Starting with some frequency ω, smaller than ω_2, and gradually increasing this frequency we will find that the amplitudes of the forced vibrations are such as are given by the ordinates of the curve $A_0A_2A_3$. This holds up to the point A_3 where an abrupt change in motion occurs. With a further increase in frequency the change in phase by 180 degrees takes place and the amplitudes are then obtained from the lower curve B_3B_4. If, after going along the curve from B_3 to B_4, we reverse the procedure and start to decrease the frequency of the disturbing force gradually, the amplitudes of the forced vibrations will be determined by the ordinates of the curve B_4B_3B. At point B an abrupt change in motion occurs, so that during a further decrease in the frequency of the disturbing force the amplitude of vibration is obtained from the curve A_2A_0. Thus a hysteresis loop $A_2A_3B_3B$ in Fig. 92 is obtained due to the instability of motion at A_3 and at B.

The curve $A_0A_2A_3BB_3B_4$ for non-linear forced motion replaces the curve in Fig. 26 relating to the case of a spring following Hooke's law. Comparing these two curves we see that instead of a vertical line of Fig. 26 corresponding to a constant critical frequency, $\omega/p = 1$, we have in Fig. 92 a curve ω_0A_3, giving frequencies of free vibrations varying with the

* Such calculations with damping can be found in the previously mentioned paper by E. V. Appleton, *loc. cit.*, p. 141.

amplitude. Also, instead of a smooth transition from oscillations in phase with the force to oscillations with 180 degrees phase difference, we have here a rather abrupt change from one motion to another at such points as A_3 and B.

In all the previous discussions it was assumed that the factor α in eq. (b) is positive, i.e., that the spring becomes stiffer as the displacement from the middle position increases. An example of such a spring is given in Fig. 75 and Fig. 77. If with the increase of the displacement the stiffness of the spring decreases as shown in Fig. 76 the factor α in eq. (b)

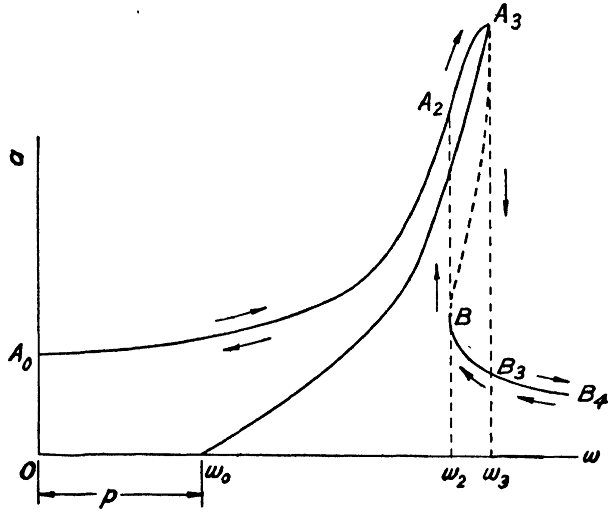

Fig. 92.

becomes negative and the frequency of the free vibrations decreases with an increase in amplitude. Proceeding as before we obtain for determining the amplitudes of the corresponding forced vibration a curve of such type as shown in Fig. 93. Starting with a small frequency of the disturbing force and gradually increasing this frequency we will find that the amplitudes of the motion are given by the ordinates of the curve $A_0A_1A_2$. At A_2 a sharp change in motion occurs. The phase of the motion changes by π and the amplitude changes from $\overline{\omega_2 A_2}$ to $\overline{\omega_2 B_2}$. With a further increase in ω, the amplitudes will be given by the curve B_2B. If we now reverse the procedure and decrease ω gradually, the amplitudes are obtained from the curve BB_2B_3, and an abrupt change in motion occurs at B_3.

It was assumed in our discussion that the spring characteristic can be represented by a smooth curve. Sometimes an abrupt change in the

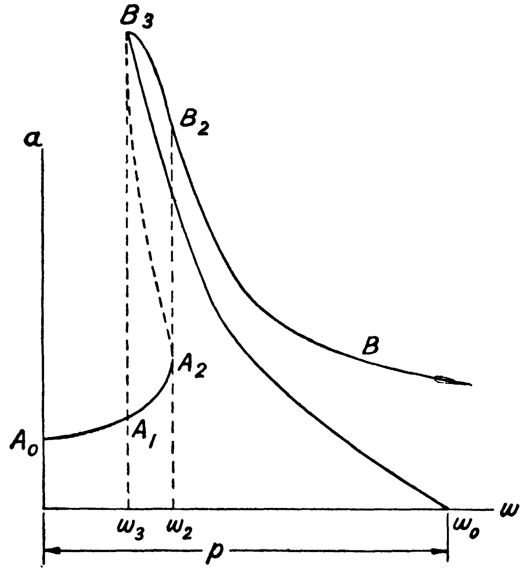

FIG. 93.

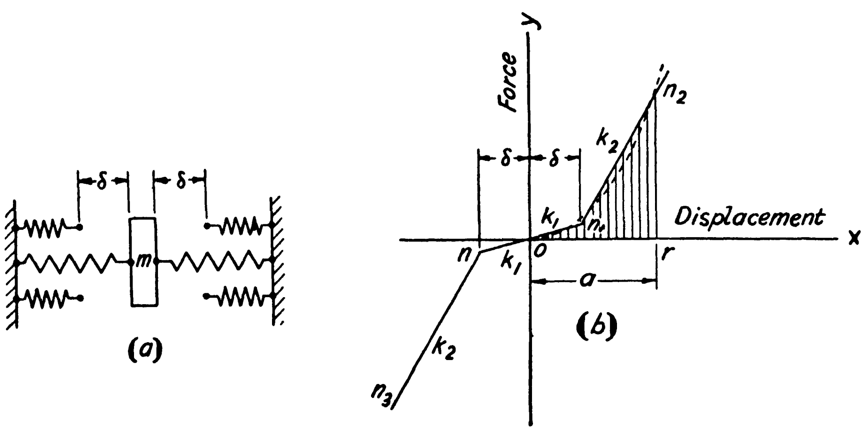

FIG. 94.

stiffness of the spring occurs during the oscillation of a system. An example of such a spring is shown in Fig. 94, *a*. When the amplitudes of vibration of the mass m are smaller than δ only two springs are in action and the spring characteristic can be represented by an inclined straight line, as the line nn_1 in Fig. 94, b. For displacements larger than δ, four more springs will be brought into action. The system becomes stiffer and its spring characteristic will now be represented by steeper lines such as lines n_1n_2 and nn_3 in Fig. 94, b. In calculating amplitudes of the steady forced vibrations of such a system we replace the broken line On_1n_2 by a cubic parabola* $y = p^2x + \alpha x^3$ and determine the parameters p^2 and α of

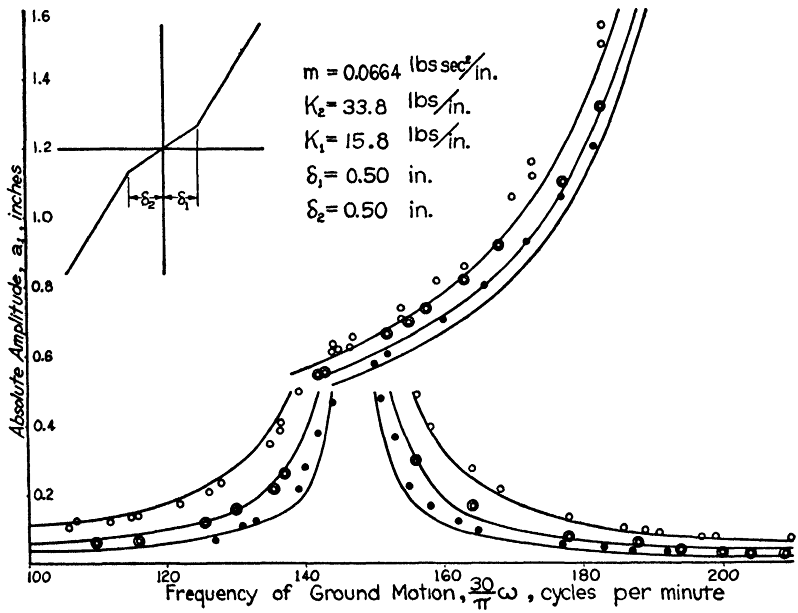

FIG. 95.

this parabola in such a manner that for $x = a$ the ordinate of the parabola is the same as the ordinate n_2r of the broken line, and that the area between the parabola and the abscissa is the same as the shaded area shown in the figure. This means that we replace the actual spring system by a fictitious spring such that the force in the spring and its potential energy at the maximum displacement a is the same as in the actual spring system. With expressions for p^2 and α, obtained in this way, we substitute in the

* This method was successfully used by L. S. Jacobsen and H. J. Jespersen, see their paper in the Journal of the Franklin Institute, Vol. 220, p. 467, 1935. The results given in our further discussion are taken from that paper.

previous eq. (*d*) and, after neglecting some small terms, a very simple equation for determining the amplitude a is obtained. Experiments show that the approximate values of the amplitude of the forced vibrations calculated in this way are in a very satisfactory agreement with experimental data. In Fig. 95 the amplitudes of the forced vibrations are plotted against the frequencies given in number of cycles per minute. Full lines give the amplitudes calculated for three different values of the disturbing force. Each set of these curves corresponds to the full line curves in Fig. 92. It may be seen that the experimental points are always very close to these lines.

The method of successive approximations, described in the previous article, can also be used for calculating amplitudes of steady forced vibration. Considering again eq. (*b*) we assume that α is small and take the solution of the equation in the following form:

$$x = \varphi_0 + \alpha\varphi_1 + \alpha^2\varphi_2 + \cdots. \tag{h}$$

We take also

$$p^2 = p_1^2 + c_1\alpha + c_2\alpha^2 + \cdots. \tag{i}$$

Substituting expressions (*h*) and (*i*) into eq. (*b*) and proceeding as explained in the previous article, we obtain for determining the functions φ_0, φ_1, $\varphi_2 \cdots$ the following system of equations

$$\begin{aligned} \ddot{\varphi}_0 + p_1^2\varphi_0 &= q \cos \omega t \\ \ddot{\varphi}_1 + p_1^2\varphi_1 &= -c_1\varphi_0 - \varphi_0^3 \\ \ddot{\varphi}_2 + p_1^2\varphi_2 &= -c_2\varphi_0 - c_1\varphi_1 - 3\varphi_0^2\varphi_1. \end{aligned} \tag{j}$$

Assuming that a steady forced vibration is built up of an amplitude a and in phase with the disturbing force we obtain the following initial conditions

$$\begin{aligned} \varphi_0(0) &= a, \qquad & \dot{\varphi}_0(0) &= 0 \\ \varphi_1(0) &= 0, \qquad & \dot{\varphi}_1(0) &= 0 \\ \varphi_2(0) &= 0, \qquad & \dot{\varphi}_2(0) &= 0 \end{aligned} \tag{k}$$

The general solution of the first of equations (*j*) is

$$\varphi_0 = C_1 \cos p_1 t + C_2 \sin p_1 t + \frac{q}{p_1^2 - \omega^2} \cos \omega t.$$

To satisfy the initial conditions given by the first row of the system (*k*) we take

$$C_1 = a - \frac{q}{p_1^2 - \omega^2}, \qquad C_2 = 0.$$

Thus

$$\varphi_0 = \left(a - \frac{q}{p_1^2 - \omega^2}\right) \cos p_1 t + \frac{q}{p_1^2 - \omega^2} \cos \omega t.$$

In order that we may have a vibration of the frequency ω, we put

$$a - \frac{q}{p_1^2 - \omega^2} = 0. \tag{l}$$

Then

$$\varphi_0 = a \cos \omega t. \tag{m}$$

Substituting φ_0 into the second of equations (j) we obtain

$$\ddot{\varphi}_1 + p_1^2\varphi_1 = - c_1 a \cos \omega t - \frac{a^3}{4}(\cos 3\omega t + 3 \cos \omega t).$$

The general solution of this equation is then:

$$\varphi_1 = C_1 \cos p_1 t + C_2 \sin p_1 t - \frac{c_1 a + \frac{3}{4} a^3}{p_1^2 - \omega^2} \cos \omega t - \frac{a^3}{4(p_1^2 - 9\omega^2)} \cos 3\omega t.$$

In order that we may have a vibration of the frequency ω and that the equation may satisfy the conditions given by the second line of the system (k), we take

$$C_1 = C_2 = 0,$$

$$\frac{c_1 a + \frac{3}{4} a^3}{p_1^2 - \omega^2} + \frac{a^3}{4(p_1^2 - 9\omega^2)} = 0. \tag{n}$$

Then

$$\varphi_1 = \frac{a^3}{4(p_1^2 - 9\omega^2)}(\cos \omega t - \cos 3\omega t),$$

so that the second approximation for x from eq. (h) is

$$x = a \cos \omega t + \frac{\alpha a^3}{4(p_1^2 - 9\omega^2)}(\cos \omega t - \cos 3\omega t). \tag{p}$$

From eq. (i) we have

$$p^2 = p_1^2 + c_1\alpha.$$

Substituting for c_1 its value from eq. (n), and using eq. (l), we obtain

$$p^2 = \frac{q}{a} + \omega^2 - \frac{3}{4}a^2\alpha - \frac{\alpha a^2}{4}\frac{1}{1 - 8\omega^2 a/q},$$

or

$$p^2 a + \alpha a^3 = q + a\omega^2 + \frac{1}{4}\alpha a^3\left(1 - \frac{1}{1 - 8\omega^2 a/q}\right). \tag{q}$$

The left side of this equation represents the force in the spring for the extreme position of the system. On the right side we have, as it can be readily shown by double differentiation of expression (p), the sum of the disturbing force and of the inertia force for the same position. Since the factor α is small, we may neglect the last term in eq. (q) and we obtain eq. (d) which was used before for approximate calculations of the amplitudes. If we keep in mind that for large vibrations the inertia force $\omega^2 a$ is usually large in comparison with the disturbing force q and neglect the second term in parenthesis of eq. (q)

as being small in comparison with unity, we find that eq. (q) coincides with eq. (g) derived before.

Substituting expressions for φ_0, φ_1, and c_1 into the third of equations (j) and proceeding as before, we can find a third approximation for x and a more accurate equation for calculating the amplitude.

Sometimes for an approximate calculation of amplitudes of forced vibrations the Ritz' method was used,* but in the case of non-linear equations the calculations of higher approximations become very complicated and the method does not represent such advantages as in the case of linear equations. Another way of calculating closer approximations for the amplitudes of forced vibrations was suggested by J. P. Den Hartog.† The approximate equation (d) was obtained by assuming a simple harmonic motion and determining its amplitude so as to satisfy equation of motion (b) for extreme positions of the vibrating system. If, instead of a simple harmonic motion, an expression containing several trigonometric terms is taken, we can determine the coefficients of these terms so as to satisfy eq. (b) not only for the extreme positions of the system but also for one or several intermediate positions.‡

In the discussion of forced vibrations we assumed that the frequency of this vibration is the same as the frequency of the disturbing force. In the case of non-linear spring characteristics, however, a harmonic force $q \cos \omega t$ may sometimes produce large vibrations of lower frequencies such as $\frac{1}{2}\omega$, $\frac{1}{3}\omega$. This phenomenon is called *sub-harmonic resonance.* The theoretical investigation of this phenomenon is a complicated one§ and we limit our discussion here to an elementary consideration which gives some explanation of the phenomena. Let us take, as an example, the case of eq. (55) discussed in the previous article. It was shown that the free vibrations in this case do not represent a simple harmonic motion and that their approximate expression contains also a higher harmonic of the third order so that for the displacement x we can take the expression

$$x = a \cos \omega t + b \cos 3\omega t. \tag{r}$$

* See G. Duffing's book, p. 130, *loc. cit.* p. 138. A similar method was recently suggested by I. K. Silverman, Journal of the Franklin Institute, Vol. 217, p. 743, 1934.

† J. P. Den Hartog, The Journal of the Franklin Institute, Vol. 216, p. 459, 1933.

‡ An exact solution of the problem for the case when the spring characteristic is represented by such a broken line as in Fig. 94, b was obtained by J. P. Den Hartog and S. J. Mikina, Trans. Am. Soc. Mech. Engrs., Vol. 54, p. 153, 1932. See also paper by J. P. Den Hartog and R. M. Heiles presented at the National Meeting of the Applied Mechanics Division, A.S.M.E., June 1936.

§ The theory of non-linear vibrations has been considerably developed in recent years, principally in connection with radio engineering. We will mention here important publications by Dr. B. van der Pol, see Phil. Mag., ser. 7, V. 3, p. 65, 1927. See also A. Andronow, Comptes Rendues, V. 189, p. 559, 1929; A. Andronow and A. Witt, C. R., v. 190, p. 256, 1930; L. Mandelstam and N. Papalexi, Zeitschr. f. Phys. Vol. 73, p. 233, 1931; N. Kryloff and N. Bogoliuboff, Schweizerische Bauzeitung, V. 103, 1934.

If there is no exciting force, this vibration, owing to unavoidable friction, will be gradually damped out. Assume now that a pulsating force $q \cos (3\,\omega t + \beta)$ is acting on the system. On the displacements (r) it will produce the following work per cycle $\tau = 2\pi/\omega$:

$$\int_0^{\tau} q \cos (3\omega t + \beta)\, \dot{x}\, dt = -\, a\omega q \int_0^{\tau} \sin \omega t \cos (3\omega t + \beta)\, dt$$
$$-\, 3b\omega q \int_0^{\tau} \sin 3\omega t \cos (3\omega t + \beta)\, dt.$$

The first term on the right side of this expression vanishes while the second term gives $3\pi\, b\, q \sin \beta$. Thus, due to the presence of the higher harmonic in expression (r), the assumed pulsating force produces work depending on the phase difference β. By a proper choice of the phase angle we may get an amount of work compensating for the energy dissipated due to damping. Thus the assumed pulsating force of frequency 3ω may maintain vibrations (r) having frequency ω and we obtain the phenomenon of subharmonic resonance.*

* The possibility of such a phenomenon in mechanical systems was indicated first by J. G. Baker, Trans. Am. Soc. Mech. Engrs., vol. 54, p. 162, 1932.

CHAPTER III

SYSTEMS WITH VARIABLE SPRING CHARACTERISTICS

27. Examples of Variable Spring Characteristics.—In the previous chapters problems were considered in which the stiffness of springs was changing with displacement. Here we will discuss cases in which the spring characteristic is varying with time.

As a first example let us consider a string AB of a length $2l$ stretched vertically and carrying at the middle a particle of mass m, Fig. 96. If x is a small displacement of the particle from its middle position, the tensile force in the string corresponding to this displacement is (see p. 116).

$$S' = S + AE\frac{x^2}{2l^2}, \tag{a}$$

where S is the tensile force in the string for static equilibrium position of the particle, A is the cross sectional area of the string and E is the modulus of elasticity of the string. Let us assume that S is very large in comparison with the change in the tensile force represented by the second term in expression (a). In such a case this second term can be neglected, $S' = S$, and the equation for motion of the particle m is:

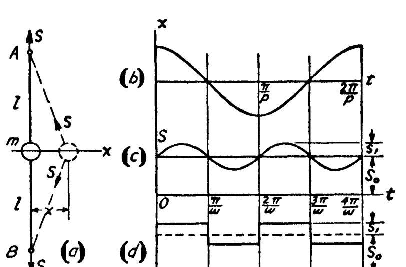

FIG. 96.

$$m\ddot{x} + \frac{2Sx}{l} = 0. \tag{b}$$

The spring characteristic in this case is defined by the quantity $2S/l$ and as long as S remains constant, equation (b) gives a simple harmonic motion of a frequency $p = \sqrt{2S/lm}$ and of an amplitude which depends on the initial conditions. If the initial displacement as well as the initial velocity of the particle are both zero, the particle remains in its middle position which is its position of stable static equilibrium.

Assume now that by some device a small steady periodic fluctuation of the tensile force S is produced such that

$$S = S_0 + S_1 \sin \omega t, \tag{c}$$

since S always remains large enough, eq. (*b*) continues to hold also in this case and we obtain a system in which the spring characteristic $2S/l$ is a periodic function of time. Without going at present into a discussion of the differential eq. (*b*), it can be seen that by a proper choice of the frequency ω of the fluctuating tension, large vibrations of the particle m can be built up. Such a condition is represented in Fig. 96, *b* and Fig. 96, *c*. The first of these curves represents displacements of the particle m when it vibrates freely under the action of a constant tension $S = S_0$, so that a complete cycle requires the time $\tau = 2\pi/p = 2\pi\sqrt{lm/2S_0}$. The second curve represents the fluctuating tension of the string which is assumed to have a circular frequency $\omega = 2p$. It is seen that during the first quarter of the cycle, when the particle m is moving from the extreme position to its middle position and the resultant of the forces S produces positive work, the average value of S is larger than S_0. During the second quarter of the cycle, when the forces S oppose the motion of the particle, their average value is smaller than S_0. Thus during each half a cycle there is a surplus of positive work produced by the tensile forces S. The result of this work is a gradual building up of the amplitude of vibration. This conclusion can be readily verified by experiment.* Furthermore, an experiment will also show that the middle position of the particle is no longer a position of stable equilibrium if a fluctuation in tensile force S of a frequency $\omega = 2p$ is maintained. A small accidental force, producing an initial displacement or an initial velocity may start vibrations which will be gradually built up as explained above.

In Fig. 96, *d* a case is represented in which the tensile force in the string is changing abruptly so that

$$S = S_0 \pm S_1. \tag{d}$$

* An example of such vibrations we have in Melde's experiment, see Phil. Mag., April, 1883. In this experiment a fine string is maintained in transverse vibrations by attaching one of its ends to the vibrating tuning-fork, the motion of the point of attachment being in the direction of the string. The period of these vibrations is double that of the fork.

By using the same reasoning as in the previous case it can be shown that changing tension S as indicated in the Fig. 96, d, will result in the production of a large vibration of the particle.

In Fig. 97 another case of the same kind is represented. On a vertical shaft is mounted a circular disc AB. Rotation of the shaft is free but its bending is confined, by the use of guiding bars nn, to the plane xy of the figure. Along most of its length the shaft has non-circular cross-section, as shown in the figure, so that its flexural rigidity in the xy plane depends on the angle of rotation. Assume first that the shaft does not rotate and in some manner its lateral vibrations in the xy plane are produced. The disc will perform a simple harmonic motion, the frequency of which depends on the flexural rigidity of the shaft. For the position of the shaft shown in the figure, flexural rigidity is a minimum and the lateral vibrations will therefore have the smallest frequency. Rotating the shaft by 90 degrees we will obtain vibrations of the highest frequency in the plane of maximum flexural rigidity. In our further discussion we will assume that the difference between the two principal rigidities is small, say not larger than ten per cent. Thus the difference between the maximum and minimum frequency of the lateral vibrations will be also small, not larger than say five per cent.

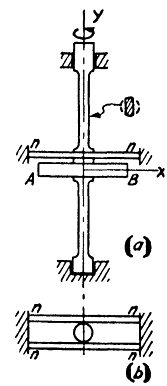

FIG. 97.

Assume now that the shaft rotates during its lateral vibrations. In such a case we obtain a vibrating system of which the spring characteristic is changing with the time, making one complete cycle during half a revolution of the shaft. By using the same kind of reasoning as in the previous case it can be shown that for a certain relation between the angular velocity ω of the shaft and the mean value p of the circular frequency of its lateral vibrations, positive work will be done on the vibrating system, and this work will result in a gradual building up of the amplitude of the lateral vibrations. Such a condition is shown by the two curves in Fig. 98. The upper curve represents the displacement-time curve for the lateral vibration of the shaft with a mean frequency p. The lower curve represents the fluctuating flexural rigidity of the shaft assuming that the shaft makes one complete revolution during one cycle of its lateral oscillations so that $\omega = p$. At the bottom of the figure the corresponding positions of rotating cross-sections of the shaft with the neutral axis n are shown. It is seen that during the first quarter of a cycle when the disc is moving from the extreme position towards the middle position and the reaction of the shaft on the disc produces positive work the flexural rigidity is larger than its average value, while during the second quarter of a cycle, when the reaction of the shaft opposes the motion of the disc, the flexural rigidity is smaller than its average value. Observing that at any instant the reaction is proportional to the corresponding flexural rigidity, it can be concluded that the positive work done during the first quarter of the cycle is numerically larger than the negative work during the second quarter. This results in a surplus of positive work during one revolution of the shaft which produces a gradual increase in the amplitude of the lateral vibrations of the shaft.

If the shaft shown in Fig. 97 is placed horizontally the action of gravity force must be taken into consideration. Assuming that the deflections

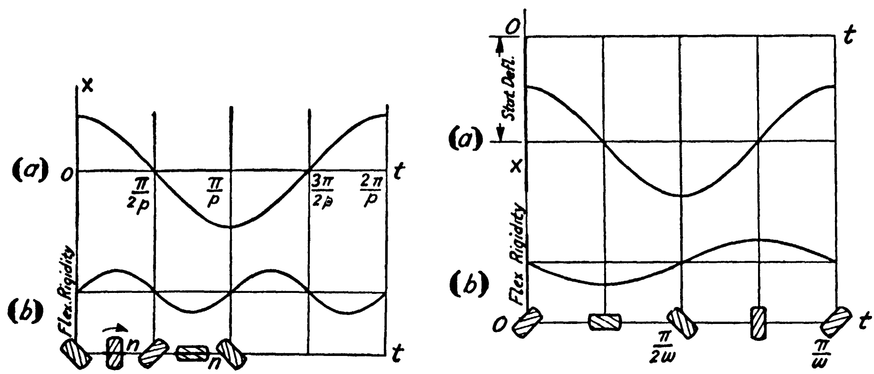

Fig. 98. Fig. 99.

due to vibrations are smaller than the statical deflection of the shaft produced by the gravity force of the disc, the displacements of the disc from the unbent axis of the shaft will always be down and can be represented during one cycle by the ordinates of the upper curve measured from the *ot* axis in Fig. 99 *a*. There are two forces acting on the disc, (1) the constant gravity force and (2) the variable reaction of the shaft on the disc which in our case has always an upward direction. The work of the gravity force during one cycle is zero, thus only the work of the reaction of the shaft should be considered. During the first half of the cycle in which the disc is mowing down the reaction opposes the motion and negative work is produced. During the second half of the cycle the reaction is acting in the direction of motion and produces positive work. If we assume, as in the previous case, that the time of one revolution of the shaft is equal to the period of the lateral vibrations and take the same curve as in Fig. 98, *b* for the fluctuating flexural rigidity, it can be seen that the total work per cycle is zero. A different conclusion will be reached if we take the angular velocity ω of the shaft two times smaller than the frequency of the lateral vibrations, so that the variation of the flexural rigidity can be represented by the lower curve in Fig. 99. It is seen that during the first half of the cycle, when the reaction is opposing the motion the flexural rigidity is smaller than its average value, and during the second half of the cycle, when the reaction is acting in the

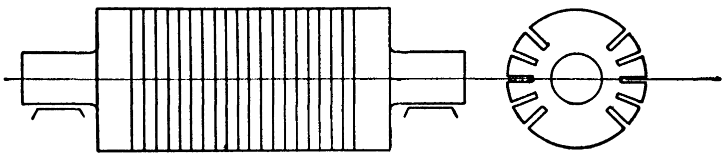

FIG. 100.

direction of motion, the flexural rigidity is larger than its average value. Thus a positive work during a cycle will be produced which will result in a building up of the amplitude of vibrations. We see that, owing to a combination of the gravity force and of the variable flexural rigidity, a large lateral vibration can be produced when the number of revolutions of the shaft per minute is only half of the number of lateral free oscillations of the shaft per minute. Such types of vibration may occur in a rotor having a variable flexural rigidity, for instance, in a two pole rotor (Fig. 100) of a turbo generator. The deflection of such a rotor under the action of its own weight varies during rotation and at a certain speed heavy vibration, due to this variable flexibility, may take place. The same kind of vibration may occur also when the non-uniformity of flexural rigidity of a rotor is due to a keyway cut in the shaft. By cutting two additional keyways, 120 degrees apart from the first, a cross-section with constant moment of inertia in all the directions will be obtained and in this way the cause of vibrations will be removed.

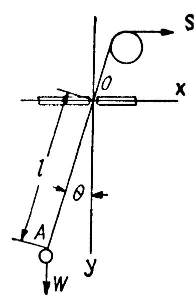

FIG. 101.

As another example let us consider a simple pendulum of variable length l (Fig. 101). By pulling the string OA with a force S, a variation in the length l of the pendulum can be produced. In order to obtain the differential equation of motion the principle of angular momentum will be applied. The momentum of the moving mass W/g can be resolved into two components, one in the direction of the string OA and another in the direction perpendicular to OA. In calculating the angular momentum about the point O only the second component equal to $(W/g)l\dot{\theta}$, must be taken into consideration. The derivative of this angular momentum with respect to the time t should be equal to the moment of the acting forces about the point O. Hence the equation

$$\frac{d}{dt}\left(\frac{W}{g}l^2\dot{\theta}\right) = -Wl\sin\theta,$$

or

$$\ddot{\theta} + \frac{2}{l}\frac{dl}{dt}\dot{\theta} + \frac{g}{l}\sin\theta = 0. \tag{57}$$

In the case of vibrations of small amplitude, θ can be substituted for $\sin\theta$ in eq. (57) and we obtain

$$\ddot{\theta} + \frac{2}{l}\frac{dl}{dt}\dot{\theta} + \frac{g}{l}\theta = 0. \tag{58}$$

When l is constant the second term on the left side of this equation vanishes and we obtain a simple harmonic motion in which g/l takes the place of the spring constant divided by the mass in eq. (b), p. 151. The variation of the length l, owing to which the second term in eq. (58) appears, may have the same effect on the vibration as the fluctuating spring stiffness discussed in the previous examples. Comparing eq. (58) with eq. (26) (see p. 33) for damped vibration, we see that the term containing the derivative dl/dt takes the place of the term representing damping in eq. (26). By an appropriate variation of the length l with time the same effect can be produced as with "*negative damping.*" In such a case a progressive accumulation of energy in the system instead of a dissipation of energy takes place and the amplitude of the oscillation of the pendulum increases with the time. It is easy to see that such an accumulation of energy results from the work done by the tensile force S during the variation in the length l of the pendulum. Various methods of varying the length l can be imagined which will result in the accumulation of energy of the vibrating system.

As an example consider the case represented in Fig. 102 in which the angular velocity $d\theta/dt$ of the pendulum and the velocity dl/dt of variation in length of the pendulum are represented as functions of the time. The period of variation of the length of the pendulum is taken half that of the vibration of the pendulum and the $d\theta/dt$ line is placed in such a manner with respect to the dl/dt line that the maximum negative damping effect coincides with the maximum speed. This means that a decrease in the length l has to be produced while the velocity $d\theta/dt$ is large and an increase

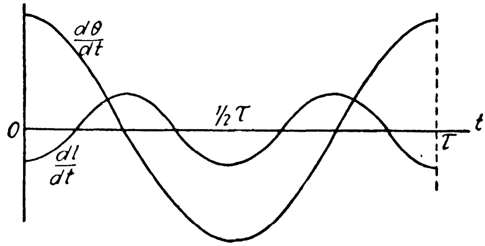

FIG. 102.

in length l while the velocity is comparatively small. Remembering that the tensile force S is working against the radial component of the weight W together with the centrifugal force, it is easy to see that in the case represented in Fig. 102 the work done by the force S during any decrease in length l will be larger than that returned during the increase in length l. The surplus of this work results in an increase in energy of vibration of the pendulum.

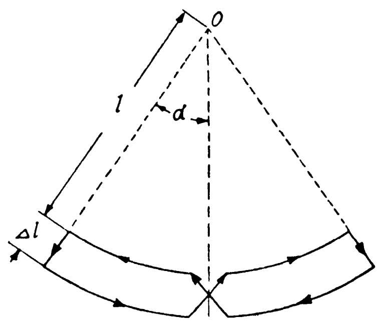

FIG. 103.

The calculation of the increase in energy of the oscillating pendulum becomes especially simple in the case shown in Fig. 103. It is assumed in this case that the length of the pendulum is suddenly decreased by the quantity Δl when the pendulum is in its middle position and is suddenly increased to the same amount when the pendulum is in its extreme positions. The trajectory of the mass W/g is shown in the figure by the full line. The mass performs two complete cycles during one oscillation of the pendulum. The work produced during the shortening of the length l of the pendulum will be

$$\left(W + \frac{W}{g}\frac{v^2}{l}\right)\Delta l.* \tag{e}$$

* In this calculation the variation in centrifugal force during the shortening of the pendulum is neglected.

Here v denotes the velocity of the mass W/g of the pendulum when in its middle position. The work returned at the extreme positions of the pendulum is

$$W \Delta l \cos \alpha. \tag{f}$$

The gain in energy during one complete oscillation of the pendulum will be

$$\Delta E = 2\left\{\left(W + \frac{W}{g}\frac{v^2}{l}\right)\Delta l - W\Delta l \cos \alpha\right\},$$

or by putting

$$v^2 = 2gl(1 - \cos \alpha),$$

we have

$$\Delta E = 6W\Delta l(1 - \cos \alpha). \tag{g}$$

Due to this increase in energy a progressive increase in amplitude of oscillation of the pendulum takes place.

In our discussion a variation of the length l of the pendulum was considered. But a similar result can be obtained if, instead of a variable length, a variable acceleration g is introduced. This can be accomplished by placing an electromagnet under the bob of the pendulum. If two cycles of the magnetic force per complete oscillation of the pendulum are produced, the surplus of energy will be put into the vibrating system during each oscillation and in this way large oscillations will be built up.

It is seen from the discussion that a vertically hanging pendulum at rest may become unstable under the action of a pulsating vertical magnetic force and vibrations, described above, can be produced if a proper timing of the magnetic action is used.* A similar effect can be produced also if a vibratory motion along the vertical axis is communicated to the suspension point of a hanging pendulum. The inertia forces of such a vertical motion are equivalent to the pulsating magnetic forces mentioned above.

If, instead of a variable spring characteristic, we have a variable oscillating mass or a variable moment of inertia of a body making torsional vibrations, the same phenomena of instability and of a gradual building up of vibrations may occur under certain conditions. Take, for example, a vertical shaft with a flywheel attached to its end (Fig. 104). The free torsional vibrations of this system will be represented by the equation

$$\frac{d}{dt}\left(I\dot{\theta}\right) + k\theta = 0, \tag{h}$$

* See Lord Rayleigh, Theory of Sound, 2nd ed., Vol. I, p. 82, 1894.

in which I is the moment of inertia of the flywheel and k is the spring constant. Let us assume now that the moment of inertia I does not remain constant and varies periodically with the time due to the harmonic motion of two symmetrically situated masses m sliding along the spokes of the wheel (Fig. 104, b). In such a case the moment of inertia can be represented by a formula

$$I = I_0(1 + \alpha \sin \omega t), \tag{i}$$

in which ω is the circular frequency of the oscillating masses m and α is a factor which we assume small in comparison with unity, so that there is only a slight fluctuation in the magnitude of the moment of inertia I. Substituting expression (i) into eq. (h), we can write this equation in the following form:

$$I_0\ddot{\theta} + \frac{I_0\alpha\omega \cos \omega t}{1 + \alpha \sin \omega t}\dot{\theta} + \frac{k}{1 + \alpha \sin \omega t}\theta = 0,$$

or, observing that α is a small quantity, we obtain

$$I_0\ddot{\theta} + I_0\alpha\omega \cos \omega t\ \dot{\theta} + k(1 - \alpha \sin \omega t)\theta = 0. \tag{j}$$

It is seen that on account of the fluctuation in the magnitude of the moment of inertia we obtain an equation (j) similar to those which we had before for the case of systems with variable spring stiffnesses. From this it can be concluded that by a proper choice of the frequency ω of the oscillating masses m large torsional vibrations of the system shown in Fig. 104 can be built up. The necessary energy for these vibrations is supplied by forces producing radial motion of the masses m. When the masses are moving toward the axis of the shaft a positive work against their centrifugal forces is produced. For a reversed motion the work is negative. If the masses be pulled towards the axis when the angular velocity of the torsional vibration and the consequent centrifugal forces are large and the motion be reversed when the centrifugal forces are small a surplus of positive work, required for building up the torsional vibrations, will be provided. Such a condition is shown in Fig. 105 in which the upper curve represents angular velocity $\dot{\theta}$ of the vibrating wheel and the lower curve represents radial displacements r of the masses m. The

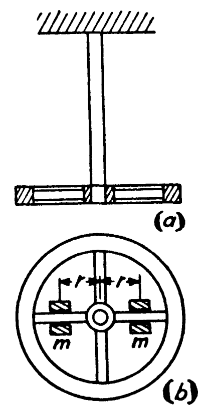

FIG. 104.

frequency of oscillation of the masses m is twice as great as the frequency of the torsional vibrations of the shaft.

If the wheel of the shaft is connected to a reciprocating mass as shown in Fig. 106 conditions similar to those just described may take place. If the upper end of the shaft is fixed and the flywheel performs small torsional vibrations such that the configuration of the system changes only little, all the masses of the system can be replaced by an equivalent disc of a constant moment of inertia (see p. 77). But if the shaft is rotating the configuration of the system is changing periodically and the equivalent disc must assume periodically varying moment of inertia.

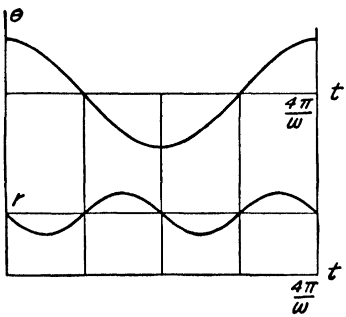

FIG. 105.

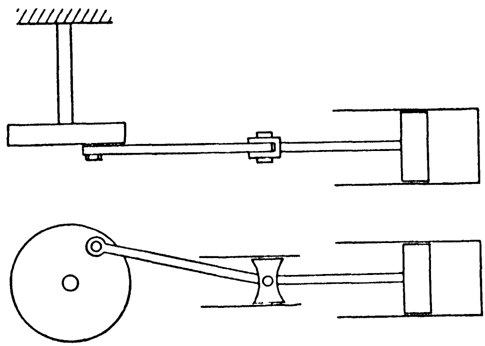

FIG. 106.

On the basis of the previous example it can be concluded that at certain angular velocities of the shaft heavy torsional vibrations in the system can be built up. These vibrations are of considerable practical importance in the case of engines with reciprocating masses.*

28. Discussion of the Equation of Vibratory Motion with Variable Spring Characteristic.—*Vibrations Without Damping.*—The differential equation of motion in the case of a variable spring characteristic can be represented in the following form if damping is neglected:

$$\ddot{x} + [p^2 + \alpha f(t)]x = 0, \qquad (a)$$

in which the term $\alpha f(t)$ is a periodical function of time defining the fluctuation of the

* This problem is discussed in the following papers. E. Trefftz, Aachener Vorträge aus dem Gebiete der Aerodynamik und verwandter Gebiete, Berlin, 1930; F. Kluge, Ingenieur-Archiv, V. 2, p. 119, 1931; T. E. Schunk, Ingenieur-Archiv, V. 2, p. 591, 1932; R. Grammel, Ingenieur-Archiv, V. 6, p. 59, 1935; R. Grammel, Zeitschr. f. angew. Math. Mech. V. 15, p. 47, 1935; N. Kotschin, Applied Mathematics and Mechanics, Vol. 2, p. 3, 1934 (Russian).

spring stiffness. In mechanical vibration problems we usually have small fluctuations of the stiffness and this term can be considered as small in comparison with p^2. When this term vanishes, eq. (*a*) coincides with that for free harmonic vibrations. In some of the examples discussed in the previous article, the fluctuation of the spring stiffness follows a sinusoidal law and the equation of motion becomes: *

$$\ddot{x} + [p^2 + \alpha \sin \omega t]x = 0. \tag{b}$$

These conditions we have, for instance, in the case of lateral vibrations of a string subjected to the action of a variable tension as shown in Fig. 96*c*.

The simplest case of a variable spring stiffness is obtained in the case represented in Fig. 96*d*, in which a ripple is superposed on the spring constant of the system. We will now discuss this later case and will show in this example how general conclusions regarding type of motion can be made from the consideration of the equation (*a*).†

The general solution of eq. (*a*) can be represented in the form ‡

$$x = C_1 e^{\mu t} \varphi(t) + C_2 e^{-\mu t} \psi(t), \tag{c}$$

where C_1 and C_2 are constants of integration, φ and ψ are periodic functions of time having the same period $\tau = 2\pi/\omega$ as the period of fluctuation of the spring characteristic and μ is a coefficient independent of t. Owing to the periodicity of the functions φ and ψ it can be seen from (*c*) that, if $x = F(t)$ is a displacement of the system at any instant t, the displacement after the elapse of an interval of time equal to the period τ will be

$$F\left(t + \frac{2\pi}{\omega}\right) = sF(t), \tag{d}$$

where s is a number depending on the magnitude of the coefficient μ. Thus if we know the motion during one cycle, the displacement at any instant of the second cycle is obtained by multiplying the corresponding displacement of the first cycle by s. In the same way the displacements of the third cycle will be obtained by using the multiplier s^2 and so on. It is seen that the type of motion depends on the magnitude of the factor s. If the absolute value of this factor is less than unity, the displacements, given by expression (*c*), will gradually die out. If the absolute value of s is larger than unity, the tendency of displacements will be to grow with time, i.e., if some initial motion to the system is given, large vibrations will be gradually built up. Thus, in this case, the motion is unstable.

* This is Mathieu's differential equation which was discussed by Mathieu in his study of vibrations of elliptical membranes, see É. Mathieu, Cours de Physique Mathematique, p. 122, Paris, 1873.

† This problem has been discussed by B. van der Pol, see Phil. Mag. Ser. 7, V. 5, p. 18, 1928. See also the paper by E. Schwerin, Zeitschr. f. Techn. Phys., V. 12, p. 104, 1931. A complete bibliography of this subject can be found in the paper by L. Mandelstam, N. Papalexi, A. Andronov, S. Chaikin and A. Witt, Exposé des recherches récentes sur les oscillations non linéaires, Technical Physics of the U.S.S.R., Vol. 2, p. 81, 1935.

‡ Floquet, Annales de l'École Normale, Vol. 1883/84.

In practical applications it is very important to know the regions in which instability takes place and a building up of large vibrations may occur. If the fluctuation of the spring stiffness consists only in a ripple superposed on the spring constant, the determination of the regions of instability can be made without much difficulty since for each half cycle the spring characteristic remains constant and the equation of simple harmonic motion can be used. Let Δ be the quantity defining the magnitude of the ripple, so that the equation of motion for the first half of a cycle, i.e., for $0 < t < \pi/\omega$, is:

$$\ddot{x} + (p^2 + \Delta)\, x = 0, \tag{e}$$

and for the second half of the cycle when $\pi/\omega < t < 2\pi/\omega$, it is

$$\ddot{x} + (p^2 - \Delta)x = 0. \tag{f}$$

Using the following notations

$$p_1 = \sqrt{p^2 + \Delta} \qquad \text{and} \qquad p_2 = \sqrt{p^2 - \Delta}. \tag{g}$$

the solutions of eqs. (*e*) and (*f*) are:

$$x_1 = C_1 \sin p_1 t + C_2 \cos p_1 t \tag{h}$$

$$x_2 = C_3 \sin p_2 t + C_4 \cos p_2 t, \tag{i}$$

where $C_1 \cdots C_4$ are the constants of integration which must be determined from the following conditions:

(1) At the end of the first half cycle ($t = \pi/\omega$) solutions (*h*) and (*i*) must agree, i.e., at this instant both solutions must give the same value for the displacement and for the velocity.

(2) At the end of a full cycle ($t = 2\pi/\omega$) the displacement and the velocity, by virtue of (*d*), must be s times as large as at the beginning. Thus the equations for determining the four constants are

$$\begin{aligned} (x_1)_{t=\pi/\omega} &= (x_2)_{t=\pi/\omega} \\ (\dot{x}_1)_{t=\pi/\omega} &= (\dot{x}_2)_{t=\pi/\omega} \\ (x_2)_{t=2\pi/\omega} &= s(x_1)_{t=0} \\ (\dot{x}_2)_{t=2\pi/\omega} &= s(\dot{x}_1)_{t=0}. \end{aligned} \tag{j}$$

Substituting for x_1 and x_2 from (*h*) and (*i*) the first of eqs. (*j*) becomes

$$C_1 \sin \frac{\pi p_1}{\omega} + C_2 \cos \frac{\pi p_1}{\omega} - C_3 \sin \frac{\pi p_2}{\omega} - C_4 \cos \frac{\pi p_2}{\omega} = 0.$$

The remaining three equations of the system (*j*) will have a similar form so that we obtain altogether four linear homogeneous equations for determining $C_1 \cdots C_4$. These equations can give solutions for the C's different from zeros only if their determinant is zero. Evaluating this determinant and equating it to zero we finally obtain the following quadratic equation for s:

$$s^2 - 2s\left(\cos \frac{\pi p_1}{\omega} \cos \frac{\pi p_2}{\omega} - \frac{p_1^2 + p_2^2}{2p_1 p_2} \sin \frac{\pi p_1}{\omega} \sin \frac{\pi p_2}{\omega}\right) + 1 = 0 \tag{k}$$

or using the notation

$$N = \cos\frac{\pi p_1}{\omega}\cos\frac{\pi p_2}{\omega} - \frac{p_1^2 + p_2^2}{2p_1p_2}\sin\frac{\pi p_1}{\omega}\sin\frac{\pi p_2}{\omega} \tag{l}$$

we obtain

$$s^2 - 2sN + 1 = 0$$

from which

$$s = N \pm \sqrt{N^2 - 1}. \tag{m}$$

It is seen that the magnitude of the factor s depends on the quantity N. If $N > 1$ one of the roots of eq. (m) is larger than unity and the vibrations will gradually build up. Hence the motion is unstable.

When N lies between $+1$ and -1 the roots of eq. (m) are complex with their moduli equal to unity. This means that there will be no tendency for the vibrations to grow so that the motion is stable.

When $N < -1$ one of the roots of eq. (m) again becomes numerically larger than unity; consequently the motion becomes unstable.

Let us now consider the physical significance of the fact that multiplier s is positive when $N > 1$, and negative when $N < -1$. Considering the displacements of the vibrating system at the ends of several consecutive cycles of the spring fluctuations, we find, from eq. (d), that in the case of positive value of s these displacements will increase and will always have the same sign. This indicates that the vibrations have the same frequency as the spring fluctuation frequency ω or they are a multiple of it. If we denote the frequency of vibrations by ω_0 we conclude that for $N > 1$ we shall have $\omega_0 = \omega$ or $\omega_0 = 2\omega$, 3ω, etc. If s is negative the displacement at the ends of the consecutive cycles of the spring fluctuations have alternating signs, which indicates that $\omega_0 = \omega/2$, $3\omega/2$, etc.

The quantity N, given by expression (l), is a function of the ratios p_1/ω and p_2/ω. By using eqs. (g) we can also represent it as a function of the ratios Δ/p^2 and p/ω. The first of these ratios gives the relative fluctuation of the spring constant and the second is the ratio of the vibration frequency of a ripple-free spring system, to the frequency of the stiffness fluctuation. If we take $(p/\omega)^2$ as abscissas and $(\Delta/p^2)(p^2/\omega^2)$ as ordinates a point in a plane for each set of values of the ratios Δ/p^2 and p/ω may be plotted and the corresponding value of N may be calculated. If such calculations have been made for a sufficient number of points, curves can be drawn that will define the transition from stable to unstable states of motion. Several curves of this kind are shown in Fig. 107,* in which the shaded areas represent the regions in which $-1 < N < 1$ (stability) and the blank area, the regions where $N > 1$ or $N < -1$ (instability). The full lines correspond to $N = +1$ and dotted lines to $N = -1$. The numbers in the regions indicate the number of oscillations of the system during one cycle, $\tau = 2\pi/\omega$, of the stiffness fluctuation.

For a given ratio Δ/p^2, i.e., for a known value of the relative fluctuation of the stiffness of the spring, the ordinates are in a constant ratio to the abscissas in Fig. 107 and we obtain an inclined line, say OA. Moving along this line we are crossing regions of stable and of unstable motions indicating that the stability of motion varies as the frequency ω of the stiffness fluctuation is changed. When ω is small we get points on the line OA far away from the origin O. As ω is gradually increased, the system passes

* See paper by B. van der Pol, loc. cit., p. 161.

through an infinite number of instability regions. Finally, as p/ω approaches the origin, the last two regions of instability are crossed, one in which the ratio p/ω is approximately unity and the other, in which p/ω is approximately one half. Experiences with such cases as discussed in the previous article indicate that these two instability regions are the most important and that large vibrations can be expected if the frequency of the stiffness fluctuation coincides with that of the free vibration * or is twice as large as that frequency. It is seen from the figure that the extents of the regions of instability such as are given by the distances $\overline{aa}$ or $\overline{bb}$ can be reduced by diminishing the slope of the line OA, i.e., by reducing the relative fluctuation of the spring stiffness. Practically such a reduction can be accomplished in the case of torsional vibrations by

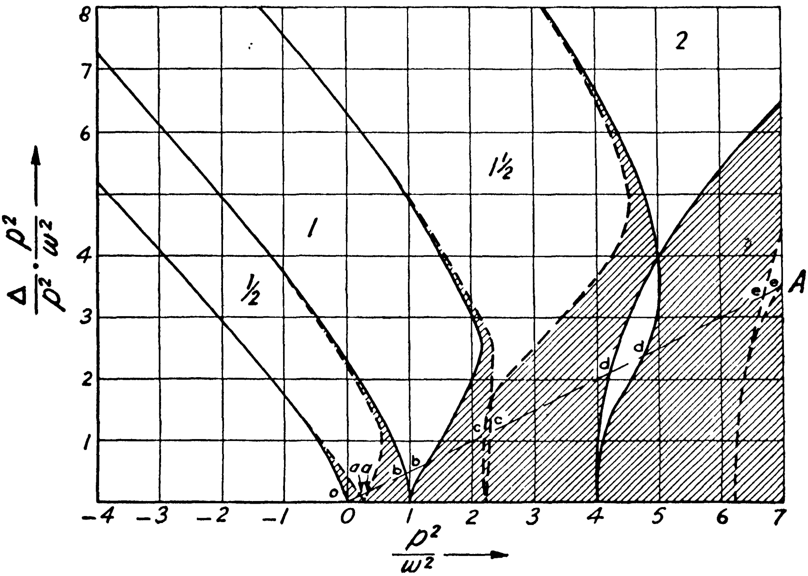

FIG. 107.

introducing flexible couplings. In this way the general flexibility of the system is increased and the relative spring fluctuation becomes smaller.

Damping is, of course, another important factor. In all our discussions damping has been neglected, thus theoretically we do not get an upper limit for the amplitude of the gradually built up vibrations. Practically this limit depends on the amount of damping, therefore, by introducing some additional friction into the flexible couplings considerable reduction in the vibrations can be effected.

Figure 107 which we used in the above discussion corresponds to the case of a rectangular ripple, but more elaborate investigations show that similar results are obtained also in the case of the sinusoidal ripple that was assumed in eq. (b).† In more general cases when the ripple superposed on the constant spring stiffness is of a more compli-

* Calculated by assuming the average value per cycle for the stiffness of the spring.

† See paper by B. van der Pol, loc. cit., p. 161.

cated form, a method of successive approximations can be used for calculating the extent of the instability regions. Taking one of the instability regions for a given stiffness fluctuation, say region $\overline{aa}$ in Fig. 107, we know that for any point in that region the numerical value of the factor s in eq. (d) is larger than unity and that the amplitude of vibration is growing. If we now consider the limiting points a, we know that at these points the numerical value of s becomes equal to unity and there is a possibility of having a steady vibratory motion. Thus the limiting points of an instability region are characterized by the fact that at these points steady vibrations of the system are possible. For the purpose of calculating the position of such points we may assume some motion of the system and investigate for what values of the frequency ω this motion becomes a steady periodic motion. These values then define the limits of the instability regions. The application of this method in studying electric locomotive vibrations will be shown in the next article.

The points to the left from the origin in Fig. 107 correspond to negative values of p^2, i.e., to negative spring constants. Such spring characteristic we may have, for instance, in the case of a pendulum. For a hanging pendulum the spring characteristic is defined by the quantity g/l, where l is the equivalent length of the pendulum. In the case of the inverted pendulum, Fig. 108, the spring characteristic is given by $-g/l$. We know that this position of equilibrium is unstable. By giving a vertical vibratory motion to the point of support A a fluctuation in the spring stiffness can be introduced (see p. 158). In such a case, as shown in Fig. 107, stability conditions can be obtained for certain frequencies of this fluctuation. Thus the pendulum will remain stable in the inverted position.

FIG. 108.

Vibrations with Damping.—As an example we take the case when damping is proportional to the velocity and the spring stiffness has a sinusoidal fluctuation of a period π/ω. The equation of motion in this case is

$$\ddot{x} + 2n\dot{x} + (p^2 - 2\alpha \sin 2\omega t)x = 0. \tag{a}$$

When α vanishes, this equation coincides with eq. (26), p. 33 for free vibrations with linear damping. From the discussion of the previous article we know that a steady vibration of a period twice as large as the period of the stiffness fluctuation can be expected in this case. We will now investigate under what conditions such a steady motion is possible. This motion will not be a simple harmonic vibration but we may represent it by a series of the period $2\pi/\omega$:

$$x = A_1 \sin \omega t + B_1 \cos \omega t + A_3 \sin 3\omega t + B_3 \cos 3\omega t + A_5 \sin 5\omega t + \cdots \tag{b}$$

and use a method of successive approximations.*

Substituting the series in eq. (a) and equating the coefficients of $\sin \omega t$, $\cos \omega t$, etc., to zero, we obtain:

$$\left.\begin{aligned}
A_1(p^2 - \omega^2) - 2n\omega B_1 - \alpha B_1 + \alpha B_3 &= 0 \\
B_1(p^2 - \omega^2) + 2n\omega A_1 - \alpha A_1 - \alpha A_3 &= 0 \\
A_3(p^2 - 9\omega^2) - 6n\omega B_3 - \alpha B_1 + \alpha B_5 &= 0 \\
B_3(p^2 - 9\omega^2) + 6n\omega A_3 + \alpha A_1 - \alpha A_5 &= 0 \\
A_5(p^2 - 25\omega^2) - 10n\omega B_5 - \alpha B_3 + \alpha B_7 &= 0 \\
B_5(p^2 - 25\omega^2) + 10n\omega A_5 + \alpha A_3 - \alpha A_7 &= 0
\end{aligned}\right\} \tag{c}$$

* Such a method of investigation was used by Lord Rayleigh, see Theory of Sound, 2nd ed., Vol. 1, p. 82, 1894.

These equations show that the coefficients A_3, B_3 are of the order α with respect to A_1, B_1; that A_5, B_5 are of order α with respect to A_3, B_3, and so on. Thus if α is small the series (b) is a rapidly converging series. The first approximation is obtained by keeping only the first two terms of the series. Omitting A_3 and B_3 in the first two of eqs. (c), we find that

$$\begin{aligned} A_1(p^2 - \omega^2) - (2n\omega + \alpha)B_1 &= 0 \\ A_1(2n\omega - \alpha) + (p^2 - \omega^2)B_1 &= 0. \end{aligned} \tag{d}$$

These equations will give solutions different from zero for A_1 and B_1 only if their determinant vanishes, whence

$$(p^2 - \omega^2)^2 = \alpha^2 - 4n^2\omega^2. \tag{e}$$

Thus, if the quantity α, defining the spring stiffness fluctuation, is known, the magnitude of the frequency ω, at which a steady motion is possible, can be found from eq. (e) which gives

$$\omega = \sqrt{p^2 - 2n^2 \pm \sqrt{(p^2 - 2n^2)^2 + \alpha^2 - p^4}}. \tag{f}$$

From eqs. (d) we also have

$$\frac{A_1}{B_1} = \frac{2n\omega + \alpha}{p^2 - \omega^2} = \frac{p^2 - \omega^2}{\alpha - 2n\omega}. \tag{g}$$

Then the first two terms of the series, representing the first approximation of the motion, can be given in the following form:

$$x_1 = C \sin(\omega t + \beta)$$

where

$$C = \sqrt{A_1^2 + B_1^2} \qquad \text{and} \qquad \beta = \arctan(B_1/A_1). \tag{h}$$

The amplitude of the vibration remains indefinite while the phase angle β can be calculated by using expression (h). If there is no damping, $2n = 0$ and we obtain

$$\omega = p\sqrt{1 \pm \frac{\alpha}{p^2}} \approx p\left(1 \pm \frac{\alpha}{2p^2}\right). \tag{i}$$

These two values of ω correspond to the two limits of the first region of instability, such as points aa in Fig. 107. Equation (e) requires that α be not less than $2n\omega$. For $\alpha < 2n\omega$ sufficient energy cannot be supplied to maintain the motion. For $\alpha = 2n\omega$ we have $\omega = p$, i.e., the frequency of the stiffness fluctuation is exactly two times larger than the free vibration frequency of the system without damping and with the assumed constant spring stiffness defined by the quantity p. The phase angle β, as may be seen from (g) and (h), is zero in this case and the relation between the motion and the spring fluctuation is such as is shown by curves (a) and (b) in Fig. 109. When $\alpha > 2n\omega$, two solutions for ω are obtained from (f) and the corresponding phase angles from (h).

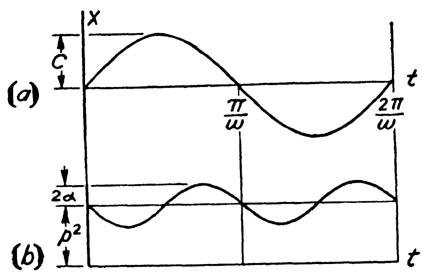

FIG. 109.

When α is much greater than $2n\omega$ the ratio A_1/B_1 in eq. (g) approaches unity and the phase angle is approximately equal to $\pm \pi/4$. For this case we therefore conclude that the curve (b) in Fig. 109 must be displaced along the horizontal axis so as to make its maximum or its minimum correspond to the zero points of the curve (a), i.e., the spring stiffness is a maximum or a minimum when the system passes through its position of equilibrium.

If a second approximation is desired we use the third and fourth of eqs. (c), from which, for small damping, we have approximately

$$A_3 = \frac{\alpha B_1}{p^2 - 9\omega^2}, \qquad B_3 = -\frac{\alpha A_1}{p^2 - 9\omega^2}. \tag{j}$$

Thus the second approximation for the motion is

$$x = C \sin (\omega t + \beta) + \frac{\alpha C}{p^2 - 9\omega^2} \cos (3\omega t + \beta). \tag{k}$$

Substituting expressions (j) in the first two of eqs. (c) we find the following more accurate equation for determining the values of ω, at which a steady motion is possible:

$$\left(p^2 - \omega^2 - \frac{\alpha^2}{p^2 - 9\omega^2}\right)^2 = \alpha^2 - 4n^2p^2 \tag{l}$$

and for the phase angle

$$\tan \beta = \left(p^2 - \omega^2 - \frac{\alpha^2}{p^2 - 9\omega^2}\right) \div (\alpha + 2n\omega).$$

Thus, by using the described method of successive approximations, we can establish the limits of the regions of instability, investigate how these limits depend on the amount of damping and determine the phase angle β. All this information is of practical interest in investigating vibrations due to fluctuation of spring stiffness.

29. Vibrations in the Side Rod Drive System of Electric Locomotives.—*General.*—One of the most important technical examples of systems with fluctuating stiffness is to be found in the case of electric locomotives with side rod drive. The flexibility of the system between the motor shaft and the driving axles depends on the position of the cranks and during uniform motion of the locomotive this is usually a complicated function of time, the period of which corresponds to one revolution of the driving axles. We have seen in the previous article that such systems of variable flexibility under certain conditions may be brought into heavy vibrations. Due to the fact that such vibrations are accompanied by a fluctuation in the angular velocity of the heavy rotating masses of the motors, large additional dynamical forces will be produced in the driving system of the locomotive. Many failures especially in the early period of electric locomotive building must be attributed to this dynamical cause.*

Variable Flexibility of Side Rod Drive.—In order to show how the flexibility of a side rod drive changes during rotation of the motor a simple example shown in Fig. 110 will now be considered. A torque M_t, acting on the rotor, is transmitted to the driving axle $0_1 0_1$ through the motor shaft 00, cranks 01 and 02 and side rods 11 and 22. Consider now the angle of rotation of the rotor with respect to the driving axle $0_1 0_1$ due to twist of the shaft 00 and due to deformation of the side rods. Let M'_t

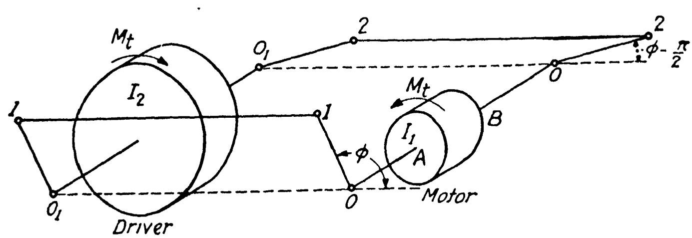

FIG. 110.

and M''_t be the moments transmitted to the driving axle through the side rods 11 and 22, respectively, then:

$$M_t = M'_t + M''_t \tag{a}$$

and if k_1 is spring characteristic for the end OA of the shaft, then the angle of rotation of the motor due to twist of the shaft will be given by

$$\Delta_1\varphi = \frac{M'_t}{k_1}. \tag{b}$$

Consider now the angle of rotation $\Delta_2\varphi$ due to compression of the side rod 11. Let,

* The most important papers dealing with vibrations in electric locomotives are:

1. "Ueber Schuttelerscheinungen in Systemen mit periodisch veranderlicher Elastizität," by Prof. E. Meissner, Schweizerische Bauzeitung, Vol. 72 (1918), p. 95.
2. "Ueber die Schüttelschwingungen des Kuppelstangantriebes," by K. E. Müller, Schweizerische Bauzeitung, Vol. 74 (1919), p. 141.
3. "Eigenschwingungen von Systemen mit periodisch veränderlicher Elastizität," by L. Dreyfus. "A. Foppl zum siebzigsten Geburtstag" (1924), p. 89.
4. A Wichert, Schüttelerscheinungen, Forschungsarbeiten, Heft 266 (1924).
5. E. E. Seefehlner, Elektrische Zugforderung, 1924.
6. A. C. Couwenhoven, Forschungsarbeiten, Heft 218, Berlin, 1919.

S_1 be compressive force in this side rod,

$\delta = \dfrac{S_1 l}{AE}$ is the corresponding compression of the side rod,

r is the radius of the crank.

Then we have

$$M'_t = S_1 r \sin \varphi, \tag{c}$$

and from a geometrical consideration (see Fig. 111),

$$\delta = r\Delta_2\varphi \sin \varphi. \tag{d}$$

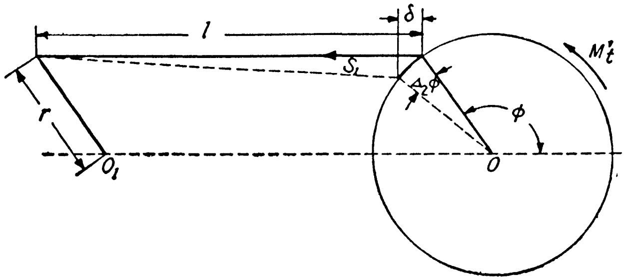

FIG. 111.

Remembering that

$$S_1 = \delta \frac{AE}{l},$$

we have from eqs. (*c*) and (*d*),

$$\Delta_2\varphi = \frac{M'_t l}{AEr^2 \sin^2 \varphi},$$

or, by letting

$$\frac{AEr^2}{l} = k_2,$$

we have

$$\Delta_2\varphi = \frac{M'_t}{k_2 \sin^2 \varphi}. \tag{e}$$

The complete angle of rotation of the motor with respect to the driving axle will be *

$$\Delta\varphi = \Delta_1\varphi + \Delta_2\varphi = M'_t\left(\frac{1}{k_1} + \frac{1}{k_2 \sin^2 \varphi}\right). \tag{f}$$

* The deformation of the side rods and of the shaft OO only are taken into consideration in this analysis.

The same angle should be obtained from a consideration of the twist of the end OB of the shaft and of compression of the side rod 22. Assuming that the arrangement is symmetrical about the longitudinal axis of the locomotive and repeating the same reasoning as above we obtain,

$$\Delta\varphi = M''_t\left(\frac{1}{k_1} + \frac{1}{k_2 \cos^2 \varphi}\right). \tag{g}$$

From equations (a), (f) and (g), we have

$$M_t = \Delta\varphi \frac{\dfrac{2}{k_1}\sin^2\varphi\cos^2\varphi + \dfrac{1}{k_2}}{\left(\dfrac{1}{k_1}\sin^2\varphi + \dfrac{1}{k_2}\right)\left(\dfrac{1}{k_1}\cos^2\varphi + \dfrac{1}{k_2}\right)}$$

$$= \Delta\varphi \frac{\dfrac{2}{k_1} + \dfrac{8}{k_2} - \dfrac{2}{k_1}\cos 4\varphi}{\dfrac{8}{k_2^2} + \dfrac{8}{k_1k_2} + \dfrac{1}{k_1^2} - \dfrac{1}{k_1^2}\cos 4\varphi}. \tag{h}$$

Putting

$$\varphi = \omega t, \qquad \frac{2}{k_1} + \frac{8}{k_2} = a, \qquad \frac{2}{k_1} = b,$$

$$\frac{8}{k_2^2} + \frac{8}{k_1k_2} + \frac{1}{k_1^2} = c, \qquad \frac{1}{k_1^2} = d,$$

we have

$$M_t = \Delta\varphi \frac{a - b\cos 4\omega t}{c - d\cos 4\omega t}. \tag{57}$$

It is seen that the flexibility of the system is a function with a period four times smaller than the period of one revolution of the shaft. In Fig. 112 the variation of the flexibility with the angle is represented graphically. For a given value of torque the angle of twist becomes maximum and equal to $M_t\left(\dfrac{1}{k_1} + \dfrac{1}{k_2}\right)$ when

$$\varphi = \omega t = 0, \frac{\pi}{2}, \pi \ldots$$

It becomes minimum and equal to $M_t\left(\frac{1}{2k_1} + \frac{1}{k_2}\right)$ when

$$\varphi = \omega t = \frac{\pi}{4}, \frac{3\pi}{4}\cdots.$$

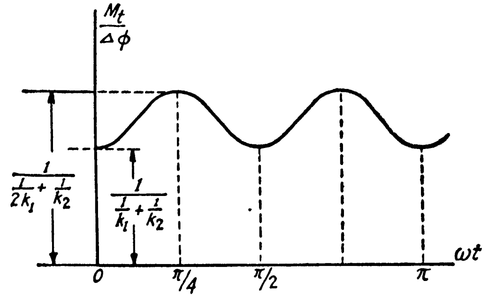

FIG. 112.

It is easy to see that the fluctuation in the flexibility of the system decreases when the rigidity of the shaft, i.e., the quantity k_1, increases. For an absolutely rigid shaft the flexibility of the system remains constant during rotation.

In our above consideration equations (a), (f) and (g) were solved analytically. The same equations can, however, be easily solved graphically.* Let AB represent to a certain scale the magnitude of the torque M_t (see Fig. 113); then by taking the end ordinates AF and BD equal to

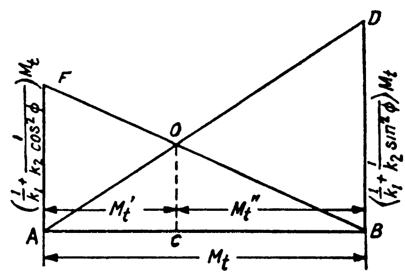

FIG. 113.

$$\left(\frac{1}{k_1} + \frac{1}{k_2 \cos^2 \varphi}\right) M_t \text{ and } \left(\frac{1}{k_1} + \frac{1}{k_2 \sin^2 \varphi}\right) M_t,$$

respectively the vertical OC through the point of intersection of the lines BF and AD will determine AC and CB, the magnitudes of the moments M'_t and M_t''.

From the figure we have also,

$$OC = M'_t\left(\frac{1}{k_1} + \frac{1}{k_2 \sin^2 \varphi}\right) = \Delta\varphi,$$

* This method was used by A. Wiechert, loc. cit., p. 168.

i.e., OC is equal to the angle of rotation of the motor due to deformation of the side rod drive produced by the torque M_t.

This graphical method is especially useful for cases in which clearances, as well as elastic, deformations are considered. Consider, for instance, the effect of the clearance between side rod and crankpin. Let a denote the magnitude of this *clearance*,* then the displacement (see Fig. 111) will be equal to the compression of the side rod together with the clearance a and we have,

$$\delta = \frac{S_1 l}{AE} + a,$$

or, by using eq. (c) and (d)

$$r\Delta_2\varphi \sin \varphi = \frac{M'_t}{r \sin \varphi} \frac{l}{AE} + a,$$

$$\Delta_2\varphi = \frac{M'_t}{k_2 \sin^2 \varphi} + \frac{a}{r \sin \varphi}.$$

The complete angle of rotation will be

$$\Delta\varphi = \Delta_1\varphi + \Delta_2\varphi = \frac{M'_t}{k_1} + \frac{M'_t}{k_2 \sin^2 \varphi} + \frac{a}{r \sin \varphi}. \tag{k}$$

In the same manner, by considering the other crank, we obtain

$$\Delta\varphi = \frac{M''_t}{k_1} + \frac{M''_t}{k_2 \cos^2 \varphi} - \frac{a}{r \cos \varphi}. \tag{l}$$

From equations (k), (l) and (a) the moments M'_t and M''_t and the angle $\Delta\varphi$ can be calculated. A graphical solution of these equations is shown in Fig. 114. AB represents, as before, the complete torque M_t. The straight lines DF and LK represent the right sides of equations (k) and (l) as linear functions of M'_t and M''_t, respectively. The point of intersection O of these two lines gives us the solution of the problem. It is easy to see that the ordinate OC is equal to the angle $\Delta\varphi$ and that the distances AC and CB are equal to the torque M'_t and M''_t, respectively.

It is seen from Fig. 114 that for the position of the cranks when

$$-\frac{a}{r \cos \varphi} = \frac{a}{r \sin \varphi} + \left(\frac{1}{k_1} + \frac{1}{k_2 \sin^2 \varphi}\right) M_t \tag{m}$$

* a denotes the difference between the radius of the bore and the radius of the pin.

M''_t becomes equal to 0. For smaller values of φ* than those given by eq. (m) the side rod 1–1 takes the complete torque and $M'_t = M_t$. In the same manner for angles larger than those obtained from the equation,

$$\frac{a}{r \sin \varphi} = -\frac{a}{r \cos \varphi} + \left(\frac{1}{k_1} + \frac{1}{k_2 \cos^2 \varphi}\right) M_t, \tag{n}$$

$M'_t = 0$, and the complete torque is taken by the side rod 2–2. By using the graphical solution (Fig. 114) within the limits determined by equations

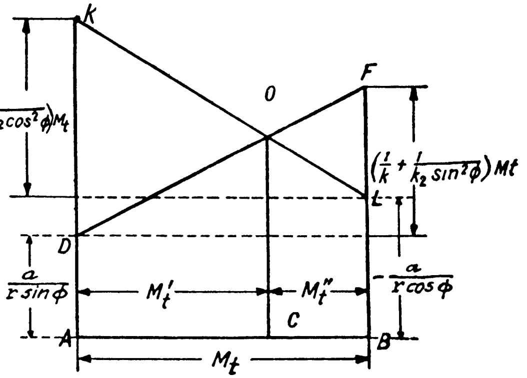

FIG. 114.

(m) and (n), and using equations (k) and (l) beyond these limits, the complete picture of the variation of the angle $\Delta\varphi$ can be obtained for $\pi/2 < \varphi < \pi$. In a similar manner other crank positions can be considered and a curve representing the variable spring characteristic $M_t/\Delta\varphi$ as a function of ωt, similar to that shown in Fig. 112, can be plotted.

Vibrations in the Side Rod Drive System.—Considering the motion of the system shown in Fig. 110,

I_1 is moment of inertia of the mass rotating about the axis 0–0,

I_2 is moment of inertia of the mass rotating about the axis 0_1–0_1,

φ_1, φ_2 are corresponding angles of rotation about 0–0 and 0_1–0_1 respectively,

$\Delta\varphi = \varphi_1 - \varphi_2$ is the angular displacement of the motor with respect to the driving axle due to deformation in the shafts and side rods,

ψ is the variable flexibility of the side rod drive, i.e., the torque necessary

* The configuration shown in Fig. 111, in which the cranks are situated in the first and second quadrants is considered here.

to produce an angular displacement $\Delta\varphi$ equal to one radian. In the particular case considered above (eq. 57, p. 170) we have

$$\psi = \frac{M_t}{\Delta\varphi} = \frac{a - b\cos 4\omega t}{c - d\cos 4\omega t}. \tag{57}$$[1]

M_t, M_r are moments of the external forces acting on the masses I_1 and I_2, respectively. When motion of the locomotive takes place a constant torque M_t acts on the mass having a moment of inertia I_1 (Fig. 110) and in opposition to this a moment $\psi(\varphi_1 - \varphi_2)$ is brought into play which represents the reaction of the elastic forces of the twisted shaft 0–0. The differential equation of motion will be

$$-I_1 \frac{d^2\varphi_1}{dt^2} + M_t - \psi(\varphi_1 - \varphi_2) = 0. \tag{a}$$

In the same manner the differential equation of motion for the second mass will be

$$-I_2 \frac{d^2\varphi_2}{dt^2} - M_r + \psi(\varphi_1 - \varphi_2) = 0. \tag{b}$$

In actual cases I_1 and I_2 represent usually the *equivalent* moments of inertia, the magnitude of which can be calculated from the consideration of the constitution of the system.

From equations (a) and (b) we have

$$\frac{d^2(\varphi_1 - \varphi_2)}{dt^2} + \psi \frac{I_1 + I_2}{I_1 I_2}(\varphi_1 - \varphi_2) = \frac{M_t}{I_1} + \frac{M_r}{I_2}.$$

Letting

$$\varphi_1 - \varphi_2 = x, \qquad \psi \frac{I_1 + I_2}{I_1 I_2} = \theta, \tag{58}$$

the following equation will be obtained:

$$\ddot{x} + \theta x = \frac{M_t}{I_1} + \frac{M_r}{I_2}, \tag{c}$$

in which θ is a certain periodical function of the time. In the case shown in Fig. 109, we have from eq. 57, p. 170.

$$\theta = \psi \frac{I_1 + I_2}{I_1 I_2} = \frac{I_1 + I_2}{I_1 I_2} \frac{a - b\cos 4\omega t}{c - d\cos 4\omega t}. \tag{59}$$

If the rigidity of the shaft is very large in comparison with that of the side rods, the quantities b and d in eq. 59 can be neglected (see p. 170) and we obtain

$$\theta = k_2 \frac{I_1 + I_2}{I_1 I_2}.$$

We arrive at a system having a constant flexibility, the circular frequency of free vibration of which can be easily found from the following equation (see p. 12)

$$p = \sqrt{\frac{k_2(I_1 + I_2)}{I_1 I_2}}. \tag{60}$$

Under the action of a variable torque M_t large vibrations in the system may arise if the period of M_t is equal to or a multiple * of the period of the free vibrations of the system. In this manner a series of critical speeds for the system will be established.

In the case of a variable flexibility the problem becomes more complicated. Instead of definite *critical speeds*, there exist definite *regions of speeds* within which large vibrations may be built up. In order to determine the limits of these critical regions an investigation of the equation

$$\ddot{x} + \theta x = 0, \tag{61}$$

representing the free vibrations of the system becomes necessary. The factor θ in this equation is a periodic function of the time depending on the variable flexibility of the system and is determined by eq. (58). Let τ denote the period of this function and $x(t)$ – a solution of eq. (61). Then, as was shown in the previous article the values of τ corresponding to the limits of the *critical regions* are those values at which one of the two following conditions is fulfilled:

$$x(t + \tau) = x(t), \tag{d}$$

$$x(t + \tau) = - x(t). \tag{e}$$

In the further discussion we will call the case (d) *a periodic solution* of the first kind and the case (e) a *periodic solution* of the second kind. It means that values τ determining the limits of the *critical regions* are those values at which eq. (61) has periodic solutions either of the first oı of the second kind.

* It is assumed that M_t is represented by a trigonometrical series (see article 18); then resonance occurs if the period of one of the terms of this series becomes equal to τ.

*Calculation of Regions of Critical Speeds.**

In the case of an electric locomotive, the fluctuation in the flexibility of the system is usually small and the regions of critical speed can be calculated by successive approximation. The procedure of these calculations will now be shown in a particular case where the function θ in the general equation,

$$\ddot{x} + \theta x = 0 \tag{62}$$

has the following form,

$$\theta = \frac{a + b\cos 2\omega t + c\cos 4\omega t}{p + q\cos 2\omega t + r\cos 4\omega t}\frac{I_1 + I_2}{I_1 I_2}. \tag{63}$$

In the case of a symmetrical arrangement, discussed above, $b = q = 0$ and we arrive at the form shown in eq. (59), p. 174.

Assuming only small fluctuations in θ during motion of the locomotive, the quantities b, c, q, and r in eq. (63) become small in comparison with a and p and, by performing the division, this equation can be represented in the following form,

$$\theta = \left\{\frac{a}{p} + \frac{b}{p}\cos 2\omega t + \frac{c}{p}\cos 4\omega t\right\}\left\{1 - \left(\frac{q}{p}\cos 2\omega t + \frac{r}{p}\cos 4\omega t\right)\right.$$
$$\left. + \left(\frac{q}{p}\cos 2\omega t + \frac{r}{p}\cos 4\omega t\right)^2 - \cdots\right\}\frac{I_1 + I_2}{I_1 I_2} \tag{a}$$

Let,

$$\frac{a}{p}\frac{I_1 + I_2}{I_1 I_2} = g_0; \quad \frac{b}{p}\frac{I_1 + I_2}{I_1 I_2} = g_1\epsilon; \quad \frac{c}{p}\frac{I_1 + I_2}{I_1 I_2} = g_2\epsilon; \quad \frac{q}{p} = g_3\epsilon; \quad \frac{r}{p} = g_4\epsilon$$

where g_1, g_2, g_3 and g_4 are quantities of the same order as g_0 and ϵ denotes a small quantity. Then by using the identity,

$$2\cos 2mt\cos 2nt = \cos 2(m + n)t + \cos 2(m - n)t, \tag{b}$$

eq. (a) can be represented in the following form,

$$\theta = \pi^2\{a_0 + \epsilon(a_1\cos 2\omega t + a_2\cos 4\omega t) + \epsilon^2(a_3\cos 2\omega t + a_4\cos 4\omega t$$
$$+ a_5\cos 6\omega t + a_6\cos 8\omega t) + \epsilon^3(a_7\cos 2\omega t + \cdots) + \cdots\}, \tag{c}$$

in which the constants a_0, a_1, a_2, $\cdots$ can be expressed by the quantities g_0, g_1, $\cdots$ given above.

It is seen now that the function θ, depending on the variable flexibility of the system, has a period

$$\tau = \frac{\pi}{\omega}, \tag{d}$$

i.e., two complete periods of θ correspond to one revolution of the crank.

In the following discussion of the differential equation (62) the angle φ, a new independent variable, instead of the time t will be introduced. This variable will be determined by the equation,

$$\varphi = \omega t, \tag{e}$$

* See Karl E. Müller, "Ueber die Schüttelschwingungen des Kuppelstangenantriebes," Dissertation der Eidgen. Techn. Hochschule in Zürich.

and will represent the angle of rotation of the crank,

$$\dot{x} = \frac{dx}{dt} = \omega \frac{dx}{d\varphi}, \qquad \ddot{x} = \frac{d^2x}{dt^2} = \omega^2 \frac{d^2x}{d\varphi^2}.$$

Substituting in eq. 62 and using eq. (d), we have

$$\frac{d^2x}{d\varphi^2} + \frac{\tau^2}{\pi^2}\theta x = 0, \tag{64}$$

in which, from eq. (c)

$$\theta = \pi^2\{a_0 + \epsilon(a_1 \cos 2\varphi + a_2 \cos 4\varphi) + \epsilon^2(a_3 \cos 2\varphi + a_4 \cos 4\varphi + a_5 \cos 6\varphi + a_6 \cos 8\varphi) + \epsilon^3(a_7 \cos 2\varphi + \cdots)\}, \tag{f}$$

i.e., the period of function θ is now equal to π.

According to the previous discussion the limits of the *critical regions* of the motion of the system correspond to those values of the period τ at which eq. (64) has *periodic* solutions of the first or second kind, i.e.,

$$x(\varphi + \pi) = x(\varphi), \tag{g}$$

or

$$x(\varphi + \pi) = -x(\varphi). \tag{g)'$$

For calculating these particular values of τ assume that τ and $x(\varphi)$ are developed in the following series,

$$\begin{aligned} \tau &= \alpha_0 + \alpha_1\epsilon + \alpha_2\epsilon^2 + \alpha_3\epsilon^3 + \cdots, \\ x(\varphi) &= x_0(\varphi) + \epsilon x_1(\varphi) + \epsilon^2 x_2(\varphi) + \cdots, \end{aligned} \tag{h}$$

in which ϵ denotes the same small quantity as in eq. (f) above.

Substituting the series (f) and (h) in eq. (64) we have

$$\frac{d^2x_0(\varphi)}{d\varphi^2} + \epsilon\frac{d^2x_1(\varphi)}{d\varphi^2} + \epsilon^2\frac{d^2x_2(\varphi)}{d\varphi^2} + \cdots + (\alpha_0 + \alpha_1\epsilon + \alpha_2\epsilon^2 + \cdots)^2 \times \\ \times \{a_0 + \epsilon(a_1 \cos 2\varphi + a_2 \cos 4\varphi) + \epsilon^2(a_3 \cos 2\varphi + \cdots)\} \times \\ \times \{x_0(\varphi) + \epsilon x_1(\varphi) + \epsilon^2 x_2(\varphi) + \cdots\} = 0.$$

Rearranging the left side of this equation in ascending powers of ϵ and equating the coefficients of every power of ϵ to zero, the following system of equations will be obtained

$$\frac{d^2x_0(\varphi)}{d\varphi^2} + \alpha_0^2 a_0 x_0(\varphi) = 0, \tag{k}$$

$$\frac{d^2x_1(\varphi)}{d\varphi^2} + \alpha_0^2 a_0 x_1(\varphi) + x_0(\varphi)\{2a_0\alpha_0\alpha_1 + \alpha_0^2(a_1 \cos 2\varphi + a_2 \cos 4\varphi)\} = 0. \tag{l}$$

Equation (k) represents a simple harmonic motion, the solution of which can be written in the form:

$$x_0 = C \cos (n\varphi - \delta_0). \tag{m}$$

In which

$$n = \sqrt{a_0\alpha_0^2} \tag{n}$$

and C and δ_0 are arbitrary constants.

In order to satisfy the conditions (g) and $(g)'$ it is necessary to take $n = 2, 4, 6, \ldots$ for *periodic* solutions of the first kind and, $n = 1, 3, 5, \ldots$ for *periodic* solutions of the second kind. Substituting this in eq. (n) and taking into account that from eq. (h) α_0 is the first approximation for the period τ, and $\pi^2\alpha_0 = \theta_0$ represents some average value of θ, we have,

$$\tau_n = \frac{n}{\sqrt{a_0}} = \frac{n\pi}{\sqrt{\theta_0}}, \tag{65}$$

in which $n = 1, 2, 3, \cdots$.

Comparing this result with the period $2\pi/\sqrt{\theta_0}$ of the natural vibrations of a system having constant flexibility $\theta = \theta_0$, it can be concluded that as a first approximation definite critical speeds are obtained instead of *critical regions of speeds*. At this critical speed the period 2τ of one revolution of the crank is equal to or a multiple of the period of the natural vibration of the system with a constant rigidity corresponding to the average value θ_0 of the function θ.

The second approximation for the solution of eq. (62) will now be obtained by substituting the first approximation (m) in equation (l). This gives

$$\frac{d^2x_1(\varphi)}{d\varphi^2} + a_0\alpha_0{}^2x_1(\varphi) = -2a_0\alpha_0\alpha_1C\cos(n\varphi - \delta_0)$$
$$-\alpha_0{}^2C\cos(n\varphi - \delta_0)(a_1\cos 2\varphi + a_2\cos 4\varphi),$$

or by using eq. (b)

$$\begin{aligned}\frac{d^2x_1(\varphi)}{d\varphi^2} + a_0\alpha_0{}^2x_1(\varphi) &= -2a_0\alpha_0\alpha_1C\cos(n\varphi - \delta_0) - \\ &-\frac{\alpha_0{}^2a_1}{2}C\{\cos[(n+2)\varphi - \delta_0] + \cos[(n-2)\varphi - \delta_0]\} - \\ &-\frac{\alpha_0{}^2a_2}{2}C\{\cos[(n+4)\varphi - \delta_0] + \cos[(n-4)\varphi - \delta_0]\}.\end{aligned} \tag{o}$$

The general solution of this equation consists of two parts: first the free vibration represented by

$$C_1\cos(n\varphi - \delta_1),$$

in which C_1 and δ_1 represent two arbitrary constants while $n = \sqrt{a_0\alpha_0{}^2}$, and another part, a forced vibration. In calculating these latter vibrations the method described on p. 103 will be used. Denoting by $R(\varphi)$ the right side of eq. (o) the forced vibration can be represented in the following form,

$$\frac{1}{n}\int_0^{\varphi} R(\xi)\sin n(\varphi - \xi)d\xi. \tag{p}$$

The terms on the right side of eq. (o) have the general form

$$N\cos[(n \pm m)\varphi - \delta].$$

Substituting this in (p) we have

$$\begin{aligned}\frac{N}{n}\int_0^{\varphi}\cos[(n \pm m)\xi - \delta]\sin n(\varphi - \xi)d\xi &= -\frac{N}{2n}\cdot\frac{1}{\pm m}\{\cos[(n \pm m)\varphi - \delta] \\ - \cos(n\varphi - \delta)\} &+ \frac{N}{2n}\frac{1}{2n \pm m}\{\cos[(n \pm m)\varphi - \delta] - \cos(n\varphi + \delta)\}.\end{aligned} \tag{q}$$

There are two exceptional cases $m = 0$ and $2n \pm m = 0$. In the first exceptional case ($m = 0$) the first term on the right side of eq. (q) becomes,

$$\frac{N}{2n}\varphi \sin(n\varphi - \delta).$$

In the second exceptional case $[(2n \pm m) = 0]$ the second term on the right side of equation (q) becomes

$$\frac{N}{2n}\varphi \sin(n\varphi + \delta).$$

After this preliminary discussion the general solution of eq. (o) can be represented in the following form,

$$
\begin{aligned}
x_1 = {} & C_1 \cos(n\varphi - \delta_1) - 2a_0\alpha_0\alpha_1 C \frac{1}{2n}\varphi \sin(n\varphi - \delta_0) \\
& - \frac{2a_0\alpha_0\alpha_1 C}{(2n)^2}[\cos(n\varphi - \delta_0) - \cos(n\varphi + \delta_0)] - \\
& - \frac{\alpha_0^2 a_1 C}{2}\left\{
\begin{aligned}
&\cos[(n+2)\varphi - \delta_0]\left(-\frac{1}{2n}\cdot\frac{1}{2} + \frac{1}{2n}\cdot\frac{1}{2n+2}\right) + \\
&\cos[(n-2)\varphi - \delta_0]\left(\frac{1}{2n}\cdot\frac{1}{2} + \frac{1}{2n}\cdot\frac{1}{2n-2}\right) + \\
&\cos(n\varphi - \delta_0)\left(\frac{1}{2n}\cdot\frac{1}{2} - \frac{1}{2n}\cdot\frac{1}{2}\right) + \\
&\cos(n\varphi + \delta_0)\left(-\frac{1}{2n}\cdot\frac{1}{2n+2} - \frac{1}{2n}\cdot\frac{1}{2n-2}\right),
\end{aligned}
\right. \qquad (r) \\
& - \frac{\alpha_0^2 a_2 C}{2\cdot 2n}\left\{
\begin{aligned}
&\cos[(n+4)\varphi - \delta_0]\left(-\frac{1}{4} + \frac{1}{2n+4}\right) + \\
&\cos[(n-4)\varphi - \delta_0]\left(\frac{1}{4} + \frac{1}{2n-4}\right) + \\
&\cos(n\varphi - \delta_0)\left(\frac{1}{4} - \frac{1}{4}\right) + \\
&\cos(n\varphi + \delta_0)\left(\frac{1}{2n+4} - \frac{1}{2n-4}\right).
\end{aligned}
\right.
\end{aligned}
$$

It is seen that all the terms of the obtained solution, except the term,

$$-2a_0\alpha_0\alpha_1 C \frac{1}{2n}\varphi \sin(n\varphi - \delta),$$

are periodical of the first or second kind; therefore the conditions (g) and (g)′ will be satisfied by putting,

$$\alpha_1 = 0.$$

In this manner the second approximation for τ will be obtained from the first of the equations (h) which approximation coincides with the first approximation. Exceptional cases only occur if $n = 1$ and $n = 2$.

In the case $n = 1$, the terms

$$-\frac{\alpha_0{}^2 a_1 C}{2}\left\{\cos\left[(n-2)\varphi - \delta_0\right]\frac{1}{2n}\cdot\frac{1}{2n-2} - \cos\left(n\varphi + \delta_0\right)\frac{1}{2n}\,\frac{1}{2n-2}\right\}$$

in the general solution (r) assume the form $\infty - \infty$ and must be replaced by the term

$$-\frac{\alpha_0{}^2 a_1 C}{2}\cdot\frac{1}{2}\cdot\varphi\sin\left(\varphi + \delta_0\right).$$

In order to make solution (r) periodical of the first or second kind it is necessary to put in this case

$$-2a_0\alpha_0\alpha_1 C\frac{1}{2}\,\varphi\sin\left(\varphi - \delta_0\right) - \frac{\alpha_0{}^2 a_1 C}{2}\,\frac{1}{2}\,\varphi\sin\left(\varphi + \delta_0\right) = 0$$

or

$$\sin\varphi\cos\delta_0\left(-a_0\alpha_0\alpha_1 - \frac{\alpha_0{}^2 a_1}{4}\right) + \cos\varphi\sin\delta_0\left(a_0\alpha_0\alpha_1 - \frac{\alpha_0{}^2 a_1}{4}\right) = 0. \tag{s}$$

There are two possibilities to satisfy this equation:

(1) $$\delta_0 = 0, \qquad -a_0\alpha_0\alpha_1 - \frac{\alpha_0{}^2 a_1}{4} = 0, \qquad \alpha_1 = -\frac{\alpha_0 a_1}{4a_0}$$

or

(2) $$\delta_0 = \frac{\pi}{2}, \qquad a_0\alpha_0\alpha_1 - \frac{\alpha_0{}^2 a_1}{4} = 0, \qquad a_1 = \frac{\alpha_0 a_1}{4a_0}.$$

Substituting the obtained values of α_1 in the first of the eqs. (h) and taking into account that, from eq. (n) for $n = 1$, $\alpha_0 = \frac{1}{\sqrt{a_0}}$, we have as the second approximation for τ_1, corresponding to the boundaries of the first critical region,

$$\tau_{1\min} = \frac{1}{\sqrt{a_0}} - \epsilon\frac{1}{\sqrt{a_0}}\cdot\frac{a_1}{4a_0},$$

$$\tau_{1\max} = \frac{1}{\sqrt{a_0}} + \epsilon\frac{1}{\sqrt{a_0}}\,\frac{a_1}{4a_0}. \tag{66}$$

It is seen that instead of a critical speed, given for $n = 1$ by eq. 65, we obtain a *critical region* between the limits $(\tau_1)_{\min}$ and $(\tau_1)_{\max}$. The extension of this region depends on the magnitude of the small quantity ϵ and it diminishes with the diminishing of the fluctuation in the flexibility of the system. It is interesting to note also that the difference in phase δ_0 between the function θ and the free vibration of the system has two definite values for the two limiting conditions, $\delta_0 = 0$ and $\delta_0 = \pi/2$. It should be noted also that the critical region considered above ($n = 1$) corresponds to the highest speed of rotation and is practically the most dangerous region.

For the case $n = 2$, i.e., for the next lower critical region, by using the same method as above, we will obtain,

$$(\tau_2)_{\substack{\min\\ \max}} = \sqrt{\frac{4}{a_0}} \mp \epsilon \sqrt{\frac{4}{a_0}} \frac{a_2}{4a_0}. \tag{67}$$

In order to obtain the critical regions corresponding to $n = 3$ and $n = 4$ the third approximation and the equation for x_2 must be considered. This equation can be obtained from the general eq. (o) in the same manner as the equations (k) and (l), used above for calculating the first and the second approximations.

By using the described method the critical regions for the equation (61) representing free vibrations in a locomotive can be established. These regions are exactly those in which heavy vibrations under the action of external forces (see eq. c, p. 174) may occur.* The investigation of actual cases shows † that the extensions of the critical regions are small and that the first approximation in which the variable rigidity is replaced by some average constant rigidity and in which the critical speeds are given by eq. 65, gives a good approximation to the actual distribution of critical speeds.

In our investigation only displacements due to elastic deformations in the system were considered. In actual conditions the problem of locomotive vibrations is much more complicated due to various kinds of clearances which always are present in the actual structure and the effect of which on the flexibility of the system have already been discussed. When the speed of a moving locomotive attains a critical region, heavy vibrations of the system may begin in which the moving masses will cross the clearances twice during every cycle.‡ The conditions will then become analogous to those shown in Fig. 81, p. 117. Such a kind of motion is accompanied with impact and is very detrimental in service. Many troublesome cases, especially in the earlier period of the building of electric locomotives, must be attributed to these vibrations. For excluding this type of vibrations a flexibility of the system must be so chosen that the operating speed of the locomotive is removed as far as possible from the critical regions. Experience shows that the detrimental effect of these vibrations can be minimized by the introduction into the system of flexible members such as, for instance, flexible gears. In this manner the fluctuation in flexibility of the system will be reduced and the extension of the critical regions of speed will be diminished. The introduction in the system of an additional damping can also be useful because it will remove the possibility of a progressive increase in the amplitude of vibrations.§

* See the paper by Prof. E. Meissner, mentioned above, p. 168.

† See the paper by K. E. Müller, mentioned above, p. 168.

‡ The possibility of the occurrence of this type of vibration can be removed to a great extent by introducing a flexible gear system.

§ Various methods of damping are discussed in the book by A. Wichert mentioned above, p. 168.

CHAPTER IV

SYSTEMS HAVING SEVERAL DEGREES OF FREEDOM

30. d'Alembert's Principle and the Principle of Virtual Displacements.—In the previous discussion of the vibration of systems having one degree of freedom d'Alembert's principle has been sometimes used. The same principle can also be applied to systems with several degrees of freedom.

As a first example the motion of a particle free in space will be considered. For determining the position of this particle three coordinates are necessary. By taking Cartesian coordinates and denoting by X, Y and Z the components of the resultant of all the forces acting on the point, the equations of equilibrium of the particle will be,

$$X = 0, \qquad Y = 0, \qquad Z = 0. \tag{68}$$

If the particle is in motion and using d'Alembert's principle the differential equations of motion can be written in the same manner as the equations of statics. It is only necessary to add the inertia force to the given external forces. The components of this force in the x, y and z directions are $-m\ddot{x}$, $-m\ddot{y}$, $-m\ddot{z}$, respectively, and the equations of motion will be

$$X - m\ddot{x} = 0, \qquad Y - m\ddot{y} = 0, \qquad Z - m\ddot{z} = 0. \tag{69}$$

If a system of several particles free in space is considered, the equations (69) should be written for every particle of the system.

Consider now systems in which the displacements of the particles constituting the system are not entirely independent but are subjected to certain constraints, which can be expressed in the form of equations between the coordinates of these points. In Fig. 115, several simple cases of such systems are represented. In the case of a spherical pendulum (Fig. 115, a) the distance of the particle m from the origin O should remain constant during motion and equal to the length l of the pendulum.

Therefore, the coordinates x, y and z of this point are no longer independent: they have to satisfy the equation*

$$x^2 + y^2 + z^2 = l^2. \tag{a}$$

In the case of a double pendulum (Fig. 115, b) the conditions of constraint will be represented by the following equations,

$$x_1{}^2 + y_1{}^2 = l_1{}^2, \tag{b}$$

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = l_2{}^2. \tag{c}$$

In the case of a connecting rod system (Fig. 115, c) the point A is moving along a circle of radius r and the point B is moving along the x

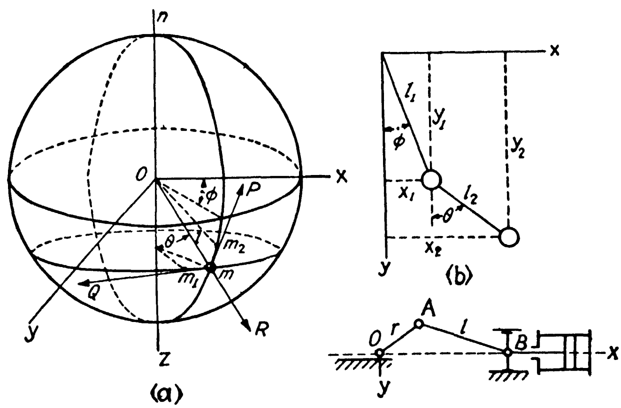

FIG. 115.

axis. The position of the system can be specified by one coordinate only, i.e., the system has only one degree of freedom.

In considering the conditions of equilibrium of such systems the *principle of virtual displacements* will be applied. This principle states that, if a system is in equilibrium, the work done by the forces on every *virtual displacement* (small possible displacement, i.e., displacement which can be performed without violating the constraints of the system) of the system must be equal to zero. For instance, in the case of a spherical pendulum denoting by X, Y and Z, the components of the resultant of

* The cases when the equations of constraint include not only the coordinates but also the velocities of the particles and the time will not be considered here. We do not consider, for instance, the oscillation of a pendulum, the length of which has to vary during the motion by a special device, i.e., the case where l is a certain function of t.

all the forces acting on the mass m, the equation of virtual displacements will be

$$X\delta x + Y\delta y + Z\delta z = 0, \tag{d}$$

in which δx, δy, δz, are the components of the virtual displacement of the point m, i.e., small changes of the coordinates x, y, z of m satisfying the condition of constraint (a). Then,

$$(x + \delta x)^2 + (y + \delta y)^2 + (z + \delta z)^2 = l^2,$$

or neglecting small quantities of higher order,

$$x\delta x + y\delta y + z\delta z = 0.$$

This equation shows that a virtual displacement is perpendicular to the bar l of the pendulum and that any small displacement of the point m on the surface of the sphere can be considered as a virtual displacement. Eq. (d) will be satisfied if the resultant of all forces acting on m be normal to the spherical surface, because only in such a case the work done by these forces on every virtual displacement will be equal to zero.

Combining now the principle of virtual displacements with d'Alembert's principle the differential equations of motion of a system with constraints can be easily obtained. For instance, in the case of a spherical pendulum, by adding the inertia force to the external forces acting on the particle m, the following general equation of motion will be obtained,

$$(X - m\ddot{x})\delta x + (Y - m\ddot{y})\delta y + (Z - m\ddot{z})\delta z = 0, \tag{70}$$

in which δx, δy, δz are components of a virtual displacement, i.e., small displacement satisfying the condition of constraint (a). In the same manner for a system consisting of n particles m_1, m_2, m_3, $\cdots$ and subjected to the action of the forces X_1, Y_1, Z_1, X_2, Y_2, Z_2, $\cdots$ the general equation of motion

$$\sum_{i=1}^{i=n} \left\{ (X_i - m_i\ddot{x}_i)\delta x_i + (Y_i - m_i\ddot{y}_i)\delta y_i + (Z_i - m_i\ddot{z}_i)\delta z_i \right\} = 0 \tag{71}$$

will be obtained, in which δx_i, δy_i, δz_i, $\cdots$ are components of virtual displacement, i.e., small displacements satisfying the conditions of constraint of the system. For instance, in the case of a double pendulum (Fig. 115, b) the virtual displacements should satisfy the conditions (see eq. b, c)

$$(x_1 + \delta x_1)^2 + (y_1 + \delta y_1) = l_1^2,$$

$$(x_2 + \delta x_2 - x_1 - \delta x_1)^2 + (y_2 + \delta y_2 - y_1 - \delta y_1)^2 = l_2^2.$$

It should be noted that X_i, Y_i and Z_i denote the components of the resultant of all the forces acting on the particle m_i, but there are several kinds of forces which do not do work on the virtual displacements; among these for instance are the reactions of connecting rods of invariable length, the reactions of fixed pins, the reactions of smooth surfaces or curves with which the moving particles are constrained to remain in contact. In the following it is assumed that in X_i, Y_i and Z_i only forces which produce work on the virtual displacements are included.

If there are no constraints and the particles of the system are completely free, the small quantities δx_i, δy_i, δz_i in eq. (71) are entirely independent and eq. (71) will be satisfied for every value of the virtual displacements only in the case that for every particle of the system the equations

$$X_i - m\ddot{x}_i = 0, \qquad Y_i - m\ddot{y}_i = 0, \qquad Z_i - m\ddot{z}_i = 0$$

are satisfied. These are the equations (69) previously obtained for the motion of a free particle.

Eq. (71) is a general equation of motion for a system of particles from which the necessary number of equations of motion, equal to the number of degrees of freedom of the system, can be derived. The derivation of these equations will be shown in Art. 32.

31. Generalized Coordinates and Generalized Forces.—In the previous article where Cartesian coordinates were used it was shown that these coordinates, as applied for describing the configuration of a system, are usually not independent. Moreover, they must satisfy certain equations of constraint, for instance, the equations (*a*), (*b*) and (*c*) of the previous article depending on the arrangement of the system. It is usually more convenient to describe the configuration of a system by means of quantities which are completely independent of each other. It is not necessary that these quantities have the dimension of a length. They may have other dimensions; for instance, it is sometimes useful to take for coordinates the angles between certain directions or the magnitudes of certain areas or certain volumes. Such independent quantities chosen for describing the configuration of a system are usually called *generalized coordinates.**

Take, for instance, the previously discussed case of a spherical pendulum (Fig. 115, *a*). The position of the pendulum will be completely determined by the two angles φ and θ shown in the figure. These two

* The terminology of "generalized coordinates," "velocities," "forces" was introduced by Thomson and Tait, Natural Philosophy, 1st edition, Oxford, 1867.

independent quantities can be taken as generalized coordinates for this case. The Cartesian coordinates of the point m can easily be expressed by the new coordinates φ and θ. Projecting the length of the pendulum Om on the coordinate axes, we have

$$\begin{aligned} x &= l \sin\theta \cos\varphi, \\ y &= l \sin\theta \sin\varphi, \\ z &= l \cos\theta. \end{aligned} \tag{a}$$

In the case of a double pendulum (Fig. 115, b) the angles φ and θ, shown in the figure, can be used as generalized coordinates and the Cartesian coordinates will be expressed by these new coordinates as follows:

$$\begin{aligned} x_1 &= l_1 \sin\varphi, & y_1 &= l_1 \cos\varphi, \\ x_2 &= l_1 \sin\varphi + l_2 \sin\theta, & y_2 &= l_1 \cos\varphi + l_2 \cos\theta. \end{aligned} \tag{b}$$

If a solid body of a homogeneous and isotropic material be subjected to a uniform external pressure p all its dimensions will be diminished in the same proportion and the new configuration will be completely determined by the change v of the volume V of the body. The quantity v can be taken as the generalized coordinate for this case.

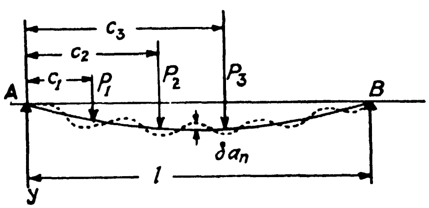

FIG. 116.

Consider now the bending of a beam supported at the ends (Fig. 116). In order to describe the configuration of this elastic system an infinitely large number of coordinates is necessary. The deflection curve can be defined by giving the deflection at every point of the beam, or we can proceed otherwise and represent the deflection curve in the form of a trigonometrical series:

$$y = a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{2\pi x}{l} + a_3 \sin\frac{3\pi x}{l} + \cdots. \tag{c}$$

The deflection curve will be completely determined if the coefficients $a_1, a_2, a_3, \cdots$ are given. These quantities may be taken for generalized coordinates in the case of the bending of a beam with supported ends.

If generalized coordinates are used for describing the configuration of a system, all the independent types of virtual displacements of the system can be obtained by giving small increases consecutively to everyone of these coordinates. By giving, for instance, a small increase $\delta\varphi$ to the angle φ in the case of the spherical pendulum a small displacement $mm_1 = l \sin \theta \delta\varphi$ along the parallel circle will be obtained. An increase of the coordinate θ by a small quantity $\delta\theta$ corresponds to a small displacement $mm_2 = l\delta\theta$ in the meridianal direction. Any other small displacement of the point m can always be resolved into two components such as mm_1 and mm_2.

In the case of bending of a beam with supported ends (Fig. 116) a small increase δa_n of any generalized coordinate a_n (see eq. *c*) involves a virtual deflection $\delta a_n \sin (n\pi x/l)$ represented in the figure by the dotted line and having n half waves. Any departure of the beam from the position of equilibrium can be obtained by superimposing such sinusoidal displacements.

Using generalized coordinates in our discussion we arrive at the notion of *generalized force.* There is a certain relation between a generalized coordinate and the corresponding generalized force, which we will explain first on simple examples. Returning again to the case of the spherical pendulum let P, Q and R represent the components of a force acting on the particle in the directions of the tangents to the meridian and to the parallel circle and in radial direction, respectively. If a small increase $\delta\varphi$ be given to the coordinate φ the point m will perform a small displacement $mm_1 = l \sin \theta\delta\varphi$ and the force acting on this point will do work equal to

$$Q\overline{mm}_1 = Ql \sin \theta\delta\varphi.$$

The factor $Ql \sin \theta$, which must be multiplied by the increase $\delta\varphi$ of the generalized coordinate φ in order to obtain the work done during the displacement $\delta\varphi$, is called the *generalized force* corresponding to the coordinate φ. In this manner we get the complete analogy with the expression $X\delta x$ of the work of the force X on the displacement δx, in the direction of the force. In the case under consideration this "force" has a simple physical meaning. It represents the moment of the forces acting on the point m about the vertical axis z. In the same manner it can be shown that the generalized force corresponding to the coordinate θ of the spherical pendulum will be represented by the moment of forces acting on the point m about the diameter perpendicular to the plane *mon.*

In the case of a body subjected to the action of a uniform hydrostatic pressure p by taking the decrease of the volume v as the generalized coordinate the corresponding generalized force will be the pressure p, because the quantity pv represents the work done by the external forces during the "displacement" v.

Let us consider now a more complicated case, namely, a beam under the action of the bending forces $P_1, P_2, \cdots$ (see Fig. 116). By taking the generalized expression (*c*) for the deflection curve and considering $a_1, a_2, a_3, \cdots$ as the generalized coordinates, the generalized force corresponding to one of these coordinates, such as a_n, will be found from a consideration of the work done by all the forces on the displacement δa_n. This displacement is represented in the figure by the dotted line.

In calculating the work produced during this displacement not only the external loads P_1, P_2 and P_3 but also internal forces of elasticity of the beam must be taken into consideration. The vertical displacements of the points of application of the loads P_1, P_2, P_3, corresponding to the increase δa_n of the coordinate a_n, will be $\delta a_n \sin (n\pi c_1/l)$, $\delta a_n \sin (n\pi c_2/l)$ and $\delta a_n \sin (n\pi c_3/l)$, respectively. The work done by P_1, P_2 and P_3 during this displacement is

$$\delta a_n\left(P_1 \sin \frac{n\pi c_1}{l} + P_2 \sin \frac{n\pi c_2}{l} + P_3 \sin \frac{n\pi c_3}{l}\right). \tag{d}$$

In order to find the work done by the forces of elasticity the expression for the potential energy of bending will be used. In the case of a beam of uniform cross section this energy is

$$V = \frac{EI}{2}\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx, \tag{e}$$

in which EI denotes the *flexural rigidity* of the beam.

Substituting in this equation the series (*c*) for y and taking into consideration that

$$\int_0^l \sin \frac{m\pi x}{l} \sin \frac{n\pi x}{l}\, dx = 0; \qquad \int_0^l \sin^2 \frac{m\pi x}{l}\, dx = \frac{l}{2},$$

where m and n denote different integer numbers, we obtain,

$$V = \frac{EI}{2}\left(\frac{a_1{}^2\pi^4}{2l^3} + \frac{a_2{}^2 2^4\pi^4}{2l^3} + \cdots\right) = \frac{EI\pi^4}{4l^3}\sum_{n=1}^{n=\infty} a_n{}^2 n^4. \tag{f}$$

The increase of the potential energy of bending due to the increase δa_n of the coordinate a_n will be, from eq. (*f*),

$$\frac{\partial V}{\partial a_n}\delta a_n = \frac{EI\pi^4}{2l^3}\, n^4 a_n \delta a_n. \tag{g}$$

This increase in potential energy is due to the work of the forces of elasticity. The work done by these forces is equal to (g) but with the opposite sign. Now, from (d) and (g) the generalized force corresponding to the coordinate a_n of the system shown in Fig. 116 will be

$$P_1 \sin \frac{n\pi c_1}{l} + P_2 \sin \frac{n\pi c_2}{l} + P_3 \sin \frac{n\pi c_3}{l} - \frac{EI\pi^4}{2l^3} n^4 a_n. \qquad (h)$$

Proceeding in the same manner we can find the generalized forces in any other case. Denoting by $q_1, q_2, q_3 \cdots$ the generalized coordinates of a system in the general case, the corresponding generalized forces Q_1, Q_2, Q_3, $\cdots$ will be found from the conditions that $Q_1 \delta q_1$ represents the work produced by all the forces during the displacement δq_1; in the same manner $Q_2 \delta q_2$ represents the work done during the displacement δq_2 and so on.

32. Lagrange's Equations.—In deriving the general equation of motion (71) by using d'Alembert's principle it was pointed out that the components δx_i, δy_i, δz_i of the virtual displacements are not independent of each other and that they must satisfy certain conditions of constraint depending on the particular arrangement of the system. A great simplification in the derivation of the equations of motion of a system may be obtained by using independent generalized coordinates and generalized forces. Let $q_1, q_2, q_3, \cdots q_k$ be the generalized coordinates of a system of n particles, with k degrees of freedom and let equations such as

$$x_i = \varphi_i(q_1, q_2 \cdots q_k); \qquad y_i = \psi_i(q_1, q_2 \cdots q_k); \qquad z_i = \theta_i(q_1, q_2 \cdots q_k) \qquad (a)$$

represent the relations between the Cartesian and the generalized coordinates. It is assumed that these equations do not contain explicitly the time t and the velocities $\dot{q}_1, \dot{q}_2, \cdots \dot{q}_k$.

In order to transform the general equation (71) to these new coordinates let us write it down in the following form,

$$\sum_{i=1}^{i=n} m_i(\ddot{x}_i \delta x_i + \ddot{y}_i \delta y_i + \ddot{z}_i \delta z_i) = \sum_{i=1}^{i=n} (X_i \delta x_i + Y_i \delta y_i + Z_i \delta z_i) \qquad (b)$$

and consider a virtual displacement corresponding to an increase δq_s of some one generalized coordinate q_s only. Then it follows at once from the definition of generalized coordinate and generalized force (see Art. 31) that the right side of eq. (b) representing the work done on the virtual displacement, is equal to

$$Q_s \delta q_s, \qquad (c)$$

where Q_s represents the generalized force corresponding to the coordinate q_s.

In order to transform the left side of eq. (b) to the new coordinates it should be noted that in the case under consideration when the coordinate q_s alone varies, the changes of the coordinates x_i, y_i, z_i will be

$$\delta x_i = \frac{\partial x_i}{\partial q_s}\,\delta q_s; \qquad \delta y_i = \frac{\partial y_i}{\partial q_s}\,\delta q_s; \qquad \delta z_i = \frac{\partial z_i}{\partial q_s}\,\delta q_s,$$

in which the symbol $\partial/\partial q_s$ denotes the partial derivative with respect to q_s and x_i, y_i, z_i are given by eqs. (a).

Then,

$$\sum_{i=1}^{i=n} m_i(\ddot{x}_i\delta x_i + \ddot{y}_i\delta y_i + \ddot{z}_i\delta z_i) = \sum_{i=1}^{i=n} m_i\left(\ddot{x}_i\frac{\partial x_i}{\partial q_s} + \ddot{y}_i\frac{\partial y_i}{\partial q_s} + \ddot{z}_i\frac{\partial z_i}{\partial q_s}\right)\delta q_s. \qquad (d)$$

Lagrange showed that this expression can be identified with certain differential operation on the expression for the kinetic energy. For this purpose we rewrite expression (d) in the following form:

$$\frac{d}{dt}\sum_{i=1}^{i=n} m_i\left(\dot{x}_i\frac{\partial x_i}{\partial q_s} + \dot{y}_i\frac{\partial y_i}{\partial q_s} + \dot{z}_i\frac{\partial z_i}{\partial q_s}\right)\delta q_s$$
$$- \sum_{i=1}^{i=n} m_i\left(\dot{x}_i\frac{d}{dt}\frac{\partial x_i}{\partial q_s} + \dot{y}_i\frac{d}{dt}\frac{\partial y_i}{\partial q_s} + \dot{z}_i\frac{d}{dt}\frac{\partial z_i}{\partial q_s}\right)\delta q_s.$$

This equation can be simplified by using the expression

$$T = \tfrac{1}{2}\sum_{i=1}^{i=n} m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)$$

for the kinetic energy of the system.

Remembering that from eqs. (a) the velocities, $\dot{x}_i$, $\dot{y}_i$, $\dot{z}_i$ can be represented as functions of the generalized coordinates q_s and the generalized velocities $\dot{q}_s$, we obtain the following expressions for the partial derivatives $\partial T/\partial \dot{q}_s$ and $\partial T/\partial q_s$

$$\frac{\partial T}{\partial \dot{q}_s} = \sum_{i=1}^{i=n} m_i\left(\dot{x}_i\frac{\partial \dot{x}_i}{\partial \dot{q}_s} + \dot{y}_i\frac{\partial \dot{y}_i}{\partial \dot{q}_s} + \dot{z}_i\frac{\partial \dot{z}_i}{\partial \dot{q}_s}\right), \qquad (e)$$

$$\frac{\partial T}{\partial q_s} = \sum_{i=1}^{i=n} m_i\left(\dot{x}_i\frac{\partial \dot{x}_i}{\partial q_s} + \dot{y}_i\frac{\partial \dot{y}_i}{\partial q_s} + \dot{z}_i\frac{\partial \dot{z}_i}{\partial q_s}\right). \qquad (f)$$

Taking into consideration that

$$\dot{x} = \frac{dx}{dt} = \frac{\partial x}{\partial q_1}\dot{q}_1 + \frac{\partial x}{\partial q_2}\dot{q}_2 + \cdots \frac{\partial x}{\partial q_k}\dot{q}_k, \qquad (g)$$

we have

$$\frac{\partial \dot{x}}{\partial \dot{q}_1} = \frac{\partial x}{\partial q_1}; \qquad \frac{\partial \dot{x}}{\partial \dot{q}_2} = \frac{\partial x}{\partial q_2}; \qquad \frac{\partial \dot{x}}{\partial \dot{q}_k} = \frac{\partial x}{\partial q_k}.$$

Now eq. (*e*) can be written as follows

$$\frac{\partial T}{\partial \dot{q}_s} = \sum_{i=1}^{i=n} m_i\left(\dot{x}_i \frac{\partial x_i}{\partial q_s} + \dot{y}_i \frac{\partial y_i}{\partial q_s} + \dot{z}_i \frac{\partial z_i}{\partial q_s}\right). \tag{h}$$

In transforming eq. (*f*) we note that

$$\frac{d}{dt}\frac{\partial x_i}{\partial q_s} = \frac{\partial^2 x_i}{\partial q_1 \partial q_s}\dot{q}_1 + \frac{\partial^2 x_i}{\partial q_2 \partial q_s}\dot{q}_2 + \cdots + \frac{\partial^2 x_i}{\partial q_k \partial q_s}\dot{q}_k,$$

or, by using (*g*),

$$\frac{d}{dt}\frac{\partial x_i}{\partial q_s} = \frac{\partial}{\partial q_s}\frac{dx_i}{dt} = \frac{\partial \dot{x}}{\partial q_s}.$$

In the same manner we have,

$$\frac{d}{dt}\frac{\partial y_i}{\partial q_s} = \frac{\partial \dot{y}}{\partial q_s}, \qquad \frac{d}{dt}\frac{\partial z_i}{\partial q_s} = \frac{\partial \dot{z}_i}{\partial q_s}.$$

Substituting this into eq. (*f*) we obtain

$$\frac{\partial T}{\partial q_s} = \sum_{i=1}^{i=n} m_i\left(\dot{x}_i \frac{d}{dt}\frac{\partial x_i}{\partial q_s} + \dot{y}_i \frac{d}{dt}\frac{\partial y_i}{\partial q_s} + \dot{z}_i \frac{d}{dt}\frac{\partial z_i}{\partial q_s}\right).$$

Now by using eq. (*h*) and (*k*) the expression (*d*) representing the left side of eq. (*b*) can be written as follows

$$\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial T}{\partial q_s}\right]\partial q_s.$$

By using for the right side of the same equation the expression (*c*) we finally obtain,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial T}{\partial q_s} = Q_s. \tag{72}$$

This is the Lagrangian form of the differential equations of motion. Such an equation can be written down for every generalized coordinate of the system so that finally the number of equations will be equal to the number of generalized coordinates, that is, the number of degrees of freedom of the system.

So far the generalized forces $Q_1, Q_2, \cdots$ have not been subject to any restriction. They may be constant forces or functions of either time, position or velocity. Consider now the particular case of forces having a potential and let V denote the potential energy of the system. Then from the condition that the work done on a virtual displacement is equal to the decrease in potential energy we have

$$Q_1\delta q_1 + Q_2\delta q_2 + Q_3\delta q_3 + \cdots = -\frac{\partial V}{\partial q_1}\delta q_1 - \frac{\partial V}{\partial q_2}\delta q_2 - \frac{\partial V}{\partial q_3}\delta q_3 - \cdots,$$

or by taking into account that the small displacements $\delta q_1, \delta q_2, \cdots$ are independent we obtain,

$$Q_1 = -\frac{\partial V}{\partial q_1}; \qquad Q_2 = -\frac{\partial V}{\partial q_2}; \qquad Q_3 = -\frac{\partial V}{\partial q_3}; \cdots$$

and the Lagrangian equation (72) takes the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial T}{\partial q_s} + \frac{\partial V}{\partial q_s} = 0. \tag{73}$$

If there are acting on the system two kinds of forces: (1) forces having a potential and (2) other forces, for which we will retain the previous notations $Q_1, Q_2, Q_3 \cdots$ Lagrange's equations become

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial T}{\partial q_s} + \frac{\partial V}{\partial q_s} = Q_s. \tag{74}$$

It was assumed in our previous discussion that the equations (a) representing the geometrical relations between the Cartesian and the generalized coordinates do not contain the time t explicitly. It can be shown, however, that Lagrange's equations retain their form also in the case when the expressions (a) vary continuously with the time, being of the type

$$x_i = \varphi_i(t, q_1, q_2 \cdots q_k); \quad y_i = \psi_i(t, q_1, q_2 \cdots q_k); \quad z_i = \theta_i(t, q_1, q_2 \cdots q_k).$$

An example of such a system will be obtained, if, for instance, we assume that the length l of a spherical pendulum shown in Fig. 115 does not remain constant but by some special arrangement is continuously varied with time.

33. Spherical Pendulum.—As an example of the application of Lagrange's equation to the solution of dynamical problems the case of the spherical pendulum (Fig. 115, a) will now be considered. By using the angles φ and θ as generalized coordinates of the particle m the velocity of m will be

$$v = \sqrt{(l\dot{\theta})^2 + (l \sin \theta \dot{\varphi})^2}$$

and the kinetic energy of the system is

$$T = \frac{m}{2} \left\{ (l\dot{\theta})^2 + (l \sin \theta \dot{\varphi})^2 \right\}. \tag{a}$$

Assuming that the weight mg is the only force acting on the mass m and proceeding as explained in article 31, we find that the generalized force corresponding to the coordinate φ is equal to zero and the force, corresponding to the coordinate θ is

$$\Theta = - mgl \sin \theta. \tag{b}$$

Using (*a*) and (*b*) the following two equations of motion will be obtained from the general equation (72)

$$\ddot{\theta} - \cos \theta \sin \theta \dot{\varphi}^2 = - \frac{g \sin \theta}{l}, \tag{c}$$

$$\sin^2 \theta \dot{\varphi} = \text{const.} = h, \text{ say.} \tag{d}$$

Several particular cases of motion will now be considered.

If the initial velocity of the particle m is in the direction of a tangent to a meridian, the path of the point will coincide with this meridian, i.e., $\dot{\varphi} = 0$, and the equation (*c*) reduces to the known equation,

$$\ddot{\theta} + \frac{g}{l} \sin \theta = 0,$$

for a simple pendulum.

The case of a conical pendulum will be obtained by assuming that the angle θ remains constant during motion, then $\dot{\varphi}$ must also be a constant, according to eq. (*d*).

Let

$$\theta = \alpha; \qquad \dot{\varphi} = \omega,$$

Then, from eq. (*c*) and (*d*)

$$\omega^2 = \frac{g}{l \cos \alpha} = \frac{h^2}{\sin^4 \alpha}, \tag{e}$$

from which the angular velocity ω and the constant h corresponding to a given angle α of a conical pendulum can be calculated.

Consider now a more complicated case where the steady motion of the mass m of the conical pendulum along a horizontal circle is slightly disturbed so that small oscillations of this mass about the horizontal circle take place. Let

$$\theta = \alpha + \xi, \tag{f}$$

where ξ denotes a small fluctuation in the angle θ during this motion. Retaining in all further calculations only the first power of the small quantity ξ we obtain

$$\sin \theta = \sin \alpha + \xi \cos \alpha; \qquad \cos \theta = \cos \alpha - \xi \sin \alpha.$$

Substituting this in eq. (c) and using eq. (d)

$$\ddot{\xi} - \frac{h^2}{\sin^3 \alpha}\left\{ \cos \alpha - \xi \left(\frac{3 \cos^2 \alpha}{\sin \alpha} + \sin \alpha \right) \right\} = - \frac{g}{l} (\sin \alpha + \xi \cos \alpha).$$

Assuming that the constant α is adjusted so that eq. (e) is satisfied, we obtain

$$\ddot{\xi} + (1 + 3 \cos^2 \alpha)\omega^2 \xi = 0,$$

from which it can be concluded that the oscillation ξ in the value of θ has the period

$$\tau = \frac{2\pi}{\omega\sqrt{1 + 3 \cos^2 \alpha}}.$$

When α is small this period approaches the value π/ω, i.e., approximately two oscillations occur for each revolution of the conical pendulum.

34. Free Vibrations. General Discussion.—If a system is disturbed from its position of stable equilibrium by an impact or by the application and sudden removal of force, the forces in the disturbed position will no longer be in equilibrium and vibrations will ensue. We consider first the case in which variable external forces are absent and *free vibrations* take place. Assuming that during vibration the system performs only small displacements, let $q_1, q_2 \cdots q_n$ be the generalized coordinates chosen in such a manner that they vanish when the system is in the position of equilibrium. Assuming now that the forces acting on the parts of the system are of the nature of elastic forces, their magnitudes will be homogeneous linear functions of the small displacements of the system, i.e., linear functions of the coordinates $q_1, q_2 \cdots q_n$. The potential energy of the system will then be a homogeneous function of the second degree of the same coordinates,

$$2V = c_{11}q_1^2 + c_{22}q_2^2 + \cdots + 2c_{12}q_1q_2 + \cdots. \tag{75}$$

The formula for the kinetic energy of the system is

$$2T = \Sigma m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)$$

or substituting here for the Cartesian coordinates their expressions in terms of the generalized coordinates (see eq. (a), article 32)

$$2T = a_{11}\dot{q}_1^2 + a_{22}\dot{q}_2^2 + \cdots + 2a_{12}\dot{q}_1\dot{q}_2 + \cdots. \tag{76}$$

In the general case the coefficients a_{11}, a_{22}, $\cdots$ will be functions of the coordinates q_1, q_2, $\cdots$ but in the case of small displacements they can be considered as constant and equal to their value at the position of equilibrium. Substituting now (75) and (76) in Lagrange's equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial T}{\partial q_s} + \frac{\partial V}{\partial q_s} = 0, \tag{a}$$

the general equations of motion will be obtained.

Consider first the case of a system with two degrees of freedom only. Then,

$$2V = c_{11}q_1^2 + c_{22}q_2^2 + 2c_{12}q_1q_2, \tag{b}$$
$$2T = a_{11}\dot{q}_1^2 + a_{22}\dot{q}_2^2 + 2a_{12}\dot{q}_1\dot{q}_2. \tag{c}$$

Since the potential energy of the system for any displacement from the position of a stable equilibrium must be positive the coefficients of the quadratic function (b) must satisfy certain conditions. Assuming $q_2 = 0$ we conclude at once that $c_{11} > 0$. In a similar manner we find that $c_{22} > 0$. Assuming now that c_{22} is different from zero we can represent expression (b) in the following form

$$2V = \frac{1}{c_{22}}[(c_{12}q_1 + c_{22}q_2)^2 + (c_{11}c_{22} - c_{12}^2)q_1^2]. \tag{b$'$}$$

To satisfy the condition that this expression is always positive we must have

$$c_{11}c_{22} - c_{12}^2 > 0,$$

since, otherwise, the expression changes the sign by passing through zero value at

$$c_{12}q_1 + c_{22}q_2 = \pm\, q_1\sqrt{-(c_{11}c_{22} - c_{12}^2)}.$$

Thus we have

$$c_{11} > 0, \qquad c_{22} > 0, \qquad c_{11}c_{22} - c_{12}^2 > 0. \tag{d}$$

In a similar way we obtain for expression (c)

$$a_{11} > 0, \qquad a_{22} > 0, \qquad a_{11}a_{22} - a_{12}^2 > 0. \tag{e}$$

Substituting expressions for V and T in eq. (a) we obtain

$$\begin{aligned} a_{11}\ddot{q}_1 + a_{12}\ddot{q}_2 + c_{11}q_1 + c_{12}q_2 &= 0, \\ a_{22}\ddot{q}_2 + a_{12}\ddot{q}_1 + c_{22}q_2 + c_{12}q_1 &= 0. \end{aligned} \tag{f}$$

The solutions of these two linear equations with constant coefficients can be taken in the following form,

$$q_1 = \lambda_1 \cos (pt + \alpha); \qquad q_2 = \lambda_2 \cos (pt + \alpha). \tag{g}$$

in which λ_1, λ_2, p and α denote constants which must be chosen so as to satisfy eqs. (f) and initial conditions.

Substituting (g) in eqs. (f) we obtain

$$\begin{aligned} \lambda_1(a_{11}p^2 - c_{11}) + \lambda_2(a_{12}p^2 - c_{12}) &= 0, \\ \lambda_1(a_{12}p^2 - c_{12}) + \lambda_2(a_{22}p^2 - c_{22}) &= 0. \end{aligned} \tag{h}$$

Eliminating now λ_1 and λ_2 we get

$$(a_{11}p^2 - c_{11})(a_{22}p^2 - c_{22}) - (a_{12}p^2 - c_{12})^2 = 0. \tag{i}$$

This equation is a quadratic in p^2 and it can be shown that it has two real positive roots.

Substituting in eq. (i) $p^2 = 0$ or $p^2 = +\infty$ and using (d) and (e) it can be concluded that the left side of this equation has a positive value. On the other hand, by taking $p^2 = (c_{11}/a_{11})$ or $p^2 = (c_{22}/a_{22})$ the left side of eq. (i) becomes negative. This means that between $p^2 = 0$ and $p^2 = +\infty$ the curve representing the left side of eq. (i) crosses the abscissa axis in two points, representing two positive roots for p^2. Let $p_1{}^2$ be one of these two roots. Substituting it in the first of eqs. (h) we have

$$\frac{\lambda_1}{a_{12}p_1{}^2 - c_{12}} = \frac{\lambda_2}{c_{11} - a_{11}p_1{}^2} = \mu_1, \text{ say.} \tag{j}$$

It is seen that for this particular root $p_1{}^2$ there is a definite ratio between the amplitudes λ_1 and λ_2 which determines the *mode* of vibration and the solution (g) becomes

$$\begin{aligned} q_1 &= \mu_1(a_{12}p_1{}^2 - c_{12}) \cos (p_1t + \alpha_1); \\ q_2 &= \mu_1(c_{11} - a_{11}p_1{}^2) \cos (p_1t + \alpha_1). \end{aligned} \tag{k}$$

Only the positive value for p_1 should be taken in this solution because the solution does not change when p_1 and α_1 change signs. The second root $p_2{}^2$ of eq. (i) gives an analogous solution with the constants μ_2 and α_2. Combining these two solutions the general solution of the eqs. (f) will be obtained.

$$
\begin{aligned}
q_1 &= \mu_1(a_{12}p_1^2 - c_{12}) \cos (p_1 t + \alpha_1) + \mu_2(a_{12}p_2^2 - c_{12}) \cos (p_2 t + \alpha_2), \\
q_2 &= \mu_1(c_{11} - a_{11}p_1^2) \cos (p_1 t + \alpha_1) + \mu_2(c_{11} - a_{11}p_2^2) \cos (p_2 t + \alpha_2),
\end{aligned} \tag{l}
$$

containing four arbitrary constants μ_1, μ_2, α_1 and α_2 which can be calculated when the initial values of the coordinates q_1 and q_2 and of the corresponding velocities $\dot{q}_1$ and $\dot{q}_2$ are given.

It is seen that in the case of a system with two degrees of freedom two *modes of vibration* are possible, corresponding to two different roots of the eq. (*i*) called the *frequency equation.* In each of these modes of vibration the generalized coordinates q_1 and q_2 are simple harmonic functions of the same period and the same phase. Each of these modes is called a *normal mode* of vibration. Its period is determined by the constants of the system and also its *type,* since the ratio between the amplitudes λ_1 and λ_2 is determined. When a system oscillates in one of the normal modes of vibration every point performs a simple harmonic motion of the same period and the same phase; all parts of the system passing simultaneously through their respective equilibrium positions.

The generalized coordinates q_1 and q_2 determining the configuration of a system can be chosen in various ways; one particular choice is especially advantageous for analytical discussion. Assume that the coordinates are chosen in such a manner that the terms containing products of the coordinates and the corresponding velocities in the expressions (*b*) and (*c*) vanish, then,

$$
\begin{aligned}
2V &= c_{11}q_1^2 + c_{22}q_2^2, \\
2T &= a_{11}\dot{q}_1^2 + a_{22}\dot{q}_2^2.
\end{aligned}
$$

The corresponding equations of motion are

$$
a_{11}\ddot{q}_1 + c_{11}q_1 = 0; \qquad a_{22}\ddot{q}_2 + c_{22}q_2 = 0;
$$

we obtain two independent differential equations so that in each normal mode of vibration only one coordinate is varying. Such coordinates are called *normal* or *principal coordinates* of the system.

In the general case of a system with n degrees of freedom substituting the expressions (75) and (76) for the potential and kinetic energies in Lagrange's equation (*a*) we obtain differential equations of motion such as

$$
\begin{aligned}
&a_{11}\ddot{q}_1 + a_{12}\ddot{q}_2 + a_{13}\ddot{q}_3 + \cdots + c_{11}q_1 + c_{12}q_2 + c_{13}q_3 + \cdots = 0, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&a_{n1}\ddot{q}_1 + a_{n2}\ddot{q}_2 + a_{n3}\ddot{q}_3 + \cdots + c_{n1}q_1 + c_{n2}q_2 + c_{n3}q_3 + \cdots = 0.
\end{aligned} \tag{m}
$$

These simultaneous differential equations are linear and of the second order with constant coefficients. Particular solutions of these equatio can be obtained by taking

$$q_1 = \lambda_1 \cos (pt + \alpha) \cdots, \qquad q_n = \lambda_n \cos (pt + \alpha).$$

Substituting in (m) we have

$$\begin{aligned} \lambda_1(a_{11}p^2 - c_{11}) + \lambda_2(a_{12}p^2 - c_{12}) \cdots + \lambda_n(a_{1n}p^2 - c_{1n}) &= 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \lambda_1(a_{n1}p^2 - c_{n1}) + \lambda_2(a_{n2}p^2 - c_{n2}) + \cdots + \lambda_n(a_{nn}p^2 - c_{nn}) &= 0. \end{aligned} \qquad ($$

Proceeding as in the case of a system with two degrees of freedom a eliminating $\lambda_1, \cdots \lambda_n$ from the equations (n) we arrive at the *frequen* *equation*

$$\Delta(p^2) = 0, \qquad (7$$

where $\Delta(p^2)$ is the determinant of eqs. (n):

$$\begin{vmatrix} (a_{11}p^2 - c_{11}), & (a_{12}p^2 - c_{12}), \cdots & (a_{1n}p^2 - c_{1n}) \\ \cdots & \cdots & \cdots \\ (a_{n1}p^2 - c_{n1}), & (a_{n2}p^2 - c_{n2}), \cdots & (a_{nn}p^2 - c_{nn}) \end{vmatrix}$$

Equation (77) is of the nth degree in p^2 and it can be shown * that the n roots of this equation are real and positive provided we have vib tion about a position of stable equilibrium of the system. Let $p_s{}^2$ be o of these roots. Substituting it in the eqs. (n) the $n - 1$ ratios

$$\lambda_1 : \lambda_2 : \lambda_3 : \cdots : \lambda_n$$

will be obtained and all amplitudes can be determined as functions one arbitrary constant, say μ_s. The corresponding solution of the eqs. (m)

$$q = \lambda_1 \cos (p_s t + \alpha_s), \cdots q_n = \lambda_n \cos (p_s t + \alpha_s). \qquad ($$

It contains two arbitrary constants μ_s and α_s and represents one of t *principal modes* of vibration. The *frequency* of this vibration, dependi on the magnitude of p_s, and the *type* of vibration, depending on the rati $\lambda_1 : \lambda_2 : \cdots$, are completely determined by the constitution of the sy tem. During this vibration all particles of the system perform simp harmonic motions of the same period $2\pi/p_s$ and of the same phase passing simultaneously through their respective equilibrium positions.

The general solution of eqs. (m) will be obtained by superimposing principal modes of vibration, such as (o), corresponding to n different roo of the frequency equation (77).

* See, for example, H. Lamb, Higher Mechanics, p. 222.

For illustrating this general theory a simple example of vibrations of a vertically stretched string with three equal and equidistant particles m will now be considered (Fig. 117, a). Assuming that lateral deflections y of the string during vibration are very small and neglecting the correspond-

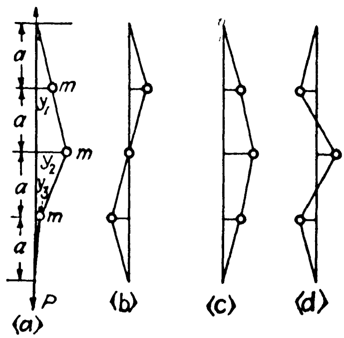

FIG. 117.

ing small fluctuations in the tensile force P, the potential energy of tension will be obtained by multiplying P with the elongation of the string.

$$V = Pa\left(\frac{1}{2}\frac{y_1^2}{a^2} + \frac{1}{2}\frac{(y_2 - y_1)^2}{a^2} + \frac{1}{2}\frac{(y_3 - y_2)^2}{a^2} + \frac{1}{2}\frac{y_3^2}{a^2}\right)$$
$$= \frac{P}{a}(y_1^2 + y_2^2 + y_3^2 - y_1y_2 - y_2y_3).$$

The kinetic energy of the system is

$$T = \frac{m}{2}(\dot{y}_1^2 + \dot{y}_2^2 + \dot{y}_3^2).$$

Substituting in Lagrange's eq. (73) we obtain:

$$\begin{aligned} m\ddot{y}_1 + \frac{P}{a}(2y_1 - y_2) &= 0, \\ m\ddot{y}_2 + \frac{P}{a}(2y_2 - y_1 - y_3) &= 0, \\ m\ddot{y}_3 + \frac{P}{a}(2y_3 - y_2) &= 0. \end{aligned} \tag{r}$$

Assuming,

$$y_1 = \lambda_1 \cos(pt - \alpha); \qquad y_2 = \lambda_2 \cos(pt - \alpha); \qquad y_3 = \lambda_3 \cos(pt - \alpha),$$

and substituting in eqs. (r) we find:

$$
\begin{aligned}
\lambda_1(p^2 - 2\beta) + \lambda_2\beta &= 0,\\
\lambda_1\beta + \lambda_2(p^2 - 2\beta) + \lambda_3\beta &= 0,\\
\lambda_2\beta + \lambda_3(p^2 - 2\beta) &= 0,
\end{aligned}
\tag{s}
$$

where,

$$\beta = \frac{P}{ma}.$$

By calculating the determinant of the eqs. (s) and equating it to zero we obtain the following frequency equation

$$(p^2 - 2\beta)(p^4 - 4p^2\beta + 2\beta^2) = 0. \tag{t}$$

Substituting the root $p^2 = 2\beta$ of this equation in eqs. (s) we have,

$$\lambda_2 = 0 \qquad \text{and} \qquad \lambda_1 = -\lambda_3;$$

the corresponding type of vibration is represented in Fig. 117, b. The two other roots, $p^2 = (2 \pm \sqrt{2})\beta$, of the same eq. ($t$), substituted in eqs. ($s$) give us

$$\lambda_1 = \lambda_3 = \pm \frac{1}{\sqrt{2}}\lambda_2.$$

The corresponding types of vibration are shown in Fig. 117, c and d. The configuration (c), where all the particles are moving simultaneously in the same direction, represents the lowest or *fundamental* type of vibration, its period being the largest. The type (d) is the highest type of vibration to which corresponds the highest frequency.

PROBLEMS

1. Investigate small vibrations of a system, Fig. 118, a, consisting of two pendula of equal masses m and length l connected by a spring at a distance h from the suspension points A and B. Masses of the spring and of the bars of the pendula can be neglected.

Solution. As generalized coordinates of the system we take the angles φ_1 and φ_2 of the pendula measured from the vertical, in a counter-clockwise direction. Then the kinetic energy of the system is:

$$T = \tfrac{1}{2} ml^2(\dot{\varphi}_1^2 + \dot{\varphi}_2^2). \tag{u}$$

The potential energy of the system consists of two parts, (1) energy due to gravity force and (2) strain energy of the spring. Considering the angles φ_1 and φ_2 as small quantities, the energy due to gravity is:

$$V_1 = mgl(1 - \cos\varphi_1) + mgl(1 - \cos\varphi_2) \approx \tfrac{1}{2} mgl(\varphi_1^2 + \varphi_2^2).$$

The spring CD for small oscillations can be assumed horizontal always. Then its elongation is $h(\sin\varphi_2 - \sin\varphi_1) \approx h(\varphi_2 - \varphi_1)$. Denoting the spring constant by k, the strain energy of the spring is

$$V_2 = \frac{k}{2}h^2(\varphi_2 - \varphi_1)^2.$$

Thus the total potential energy of the system is

$$V = \tfrac{1}{2}mgl(\varphi_1^2 + \varphi_2^2) + \tfrac{1}{2}kh^2(\varphi_2 - \varphi_1)^2$$
$$= \tfrac{1}{2}[(mgl + kh^2)(\varphi_1^2 + \varphi_2^2) - 2kh^2\varphi_1\varphi_2]. \quad (v)$$

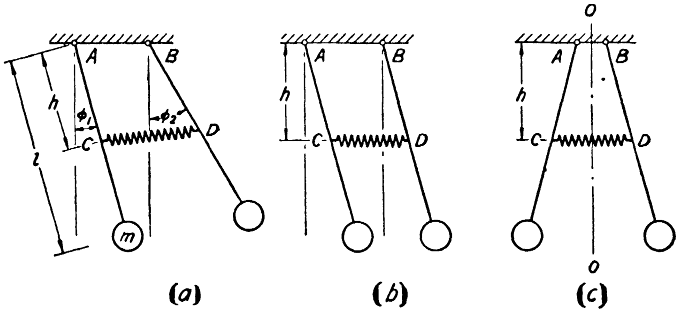

FIG. 118.

Comparing (u) and (v) with the general expressions (b) and (c) for potential and kinetic energy, we find that in the case under consideration

$$\begin{aligned} a_{11} &= a_{22} = ml^2, & a_{12} &= 0, \\ c_{11} &= c_{22} = mgl + kh^2, & c_{12} &= -kh^2. \end{aligned} \quad (w)$$

Substituting these values in the frequency equation (i) we obtain

$$(ml^2p^2 - mgl - kh^2)^2 - k^2h^4 = 0.$$

From this equation we find the two roots for p^2

$$p_1^2 = \frac{g}{l}, \quad p_2^2 = \frac{g}{l} + \frac{2kh^2}{ml^2}.$$

The ratios of the amplitudes for the corresponding two modes of vibrations, from eq. (j), are:

$$\left(\frac{\varphi_1}{\varphi_2}\right)_1 = \frac{a_{12}p_1^2 - c_{12}}{c_{11} - a_{11}p_1^2} = \frac{kh^2}{mgl + kh^2 - mgl} = 1,$$

$$\left(\frac{\varphi_1}{\varphi_2}\right)_2 = \frac{a_{12}p_2^2 - c_{12}}{c_{11} - a_{11}p_2^2} = \frac{kh^2}{mgl + kh^2 - mgl - 2kh^2} = -1.$$

These two modes of vibration are shown in Fig. 118, b and c. In the first mode of vibration the pendula have the same amplitude and their vibrations are in phase. There is no force in the spring so that the frequency of vibration is the same as for a simple pendulum. In the second mode of vibration, Fig. 118, c, there is a phase difference of 180 degrees in the oscillation of the two pendula and the spring comes into play which means that a higher frequency is obtained. This later frequency can be found in an elementary way, without using Lagrange's equations, if we observe that the configuration of the system is symmetrical with respect to the vertical axis 0–0. Considering the motion of one of the two pendula and noting that the force in the spring is $2k\varphi h$, the principle of moment of momentum with respect to the suspension point of the pendulum gives

$$\frac{d}{dt}(m\dot{\varphi}l^2) = -(mgl\varphi + 2k\varphi h^2),$$

from which the frequency p_2, calculated above, results. Having found the principal modes of vibration, we may write the general solution by superposing these two vibrations taking each mode of vibration with its proper amplitude and its proper phase angle. Thus we obtain the following general expressions for each coordinate

$$\begin{aligned} \varphi_1 &= a_1 \sin(p_1 t + \alpha_1) + a_2 \sin(p_2 t + \alpha_2), \\ \varphi_2 &= a_1 \sin(p_1 t + \alpha_1) - a_2 \sin(p_2 t + \alpha_2), \end{aligned}$$

in which the constants a_1, a_2, α_1 and α_2 are to be determined from the initial conditions.

Assume, for instance, that at the initial instant ($t = 0$) the pendulum to the left has the angle of inclination φ_0 while the pendulum to the right is vertical; moreover the initial velocities of both pendula are zero. Then

$$(\varphi_1)_{t=0} = \varphi_0, \quad (\varphi_2)_{t=0} = 0, \quad (\dot{\varphi}_1)_{t=0} = (\dot{\varphi}_2)_{t=0} = 0.$$

These conditions are satisfied in the general solution by taking

$$a_1 = a_2 = \tfrac{1}{2}\varphi_0 \quad \text{and} \quad \alpha_1 = \alpha_2 = \tfrac{1}{2}\pi.$$

Then

$$\varphi_1 = \frac{\varphi_0}{2}(\cos p_1 t + \cos p_2 t) = \varphi_0 \cos\frac{p_1 - p_2}{2}t \cos\frac{p_1 + p_2}{2}t,$$

$$\varphi_2 = \frac{\varphi_0}{2}(\cos p_1 t - \cos p_2 t) = \varphi_0 \sin\frac{p_2 - p_1}{2}t \cos\frac{p_1 + p_2}{2}t.$$

If the two frequencies p_1 and p_2 are close to one another, each coordinate contains a product of two trigonometric functions, one of low frequency $(p_1 - p_2)/2$ and the other of high frequency $(p_1 + p_2)/2$. Thus a phenomenon of beating (see p. 17) takes place. At the beginning we have vibrations of the pendulum to the left. Gradually its amplitude decreases, while the amplitude of the pendulum to the right increases and after an interval of time $\pi/(p_1 - p_2)$ only the second pendulum will be in motion. Immediately thereupon the vibration of the first pendulum begins to increase and so on.

2. Investigate the small vibrations of a double pendulum consisting of two rigid bodies suspended at A and hinged at B, Fig. 119.

Solution. Taking, for coordinates, the angles of inclination φ_1 and φ_2, which the bodies are making with their vertical positions of equilibrium, and using notations W_1 and W_2 for the weights of the bodies, applied at the centers of gravity C_1 and C_2, and I_1

and I_2 for the moments of inertia of the upper body with respect to A and of the lower body with respect to C_2 respectively, the kinetic energy of the upper body is

$$T_1 = \tfrac{1}{2} I_1 \dot{\varphi}_1^2.$$

The kinetic energy of the lower body consists of two parts, (1) owing to the rotation of the body with respect to its center of gravity C_2, and (2) owing to the linear velocity v of this center, which is equal to the geometrical sum of the velocity $l\dot{\varphi}_1$ of the hinge B plus the rotational velocity $h_2\dot{\varphi}_2$ with respect to the hinge. Thus, from Fig. 119 we find

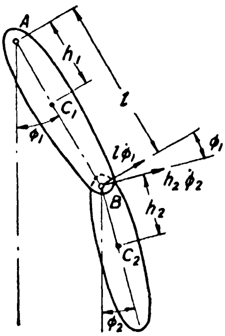

FIG. 119.

$$v^2 = l^2\dot{\varphi}_1^2 + h_2^2\dot{\varphi}_2^2 + 2h_2 l\dot{\varphi}_1\dot{\varphi}_2 \cos(\varphi_1 - \varphi_2)$$

and

$$T_2 = \tfrac{1}{2} I_2\dot{\varphi}_2^2 + \frac{W_2}{2g} v^2.$$

Assuming that the angles φ_1 and φ_2 are small and taking $\cos(\varphi_1 - \varphi_2) \approx 1$, we obtain the following expression for the total kinetic energy

$$T = \tfrac{1}{2}\left[\left(I_1 + \frac{W_2 l^2}{g}\right)\dot{\varphi}_1^2 + \left(I_2 + \frac{h_2^2 W_2}{g}\right)\dot{\varphi}_2^2 + 2h_2 l\frac{W_2}{g}\dot{\varphi}_1\dot{\varphi}_2\right].$$

The potential energy of the system is entirely due to gravity forces. Observing that the vertical displacements of the centers C_1 and C_2 are

$$h_1(1 - \cos\varphi_1) \approx \frac{h_1\varphi_1^2}{2} \quad \text{and} \quad l(1 - \cos\varphi_1) + h_2(1 - \cos\varphi_2) \approx \frac{l\varphi_1^2}{2} + \frac{h_2\varphi_2^2}{2},$$

the expression for V becomes

$$V = \tfrac{1}{2}[W_1 h_1\varphi_1^2 + W_2(l\varphi_1^2 + h_2\varphi_2^2)] = \tfrac{1}{2}[(W_1h_1 + W_2 l)\varphi_1^2 + W_2h_2\varphi_2^2].$$

Comparing the expressions for V and T with expressions (b) and (c) we find for our problem

$$a_{11} = I_1 + \frac{W_2}{g}l^2, \quad a_{22} = I_2 + \frac{W_2}{g}h_2^2, \quad a_{12} = h_2 l\frac{W_2}{g},$$
$$c_{11} = W_1h_1 + W_2 l, \quad c_{22} = W_2h_2, \quad c_{12} = 0.$$

The frequency equation (i) becomes

$$\left[\left(I_1 + \frac{W_2}{g}l^2\right)p^2 - (W_1h_1 + W_2 l)\right]\left[\left(I_2 + \frac{W_2}{g}h_2^2\right)p^2 - W_2h_2\right] - h_2^2 l^2\left(\frac{W_2}{g}\right)^2 p^4 = 0$$

To simplify the writing we introduce the following notations:

$$\frac{c_{11}}{a_{11}} = \frac{W_1h_1 + W_2 l}{I_1 + \dfrac{W_2}{g}l^2} = n_1^2, \quad \frac{c_{22}}{a_{22}} = \frac{W_2h_2}{I_2 + \dfrac{W_2}{g}h_2^2} = n_2^2,$$

$$\frac{W_2^2h_2^2l^2}{g^2\left(I_1 + \dfrac{W_2}{g}l^2\right)\left(I_2 + \dfrac{W_2}{g}h_2^2\right)} = n_3^2,$$

and the frequency equation will be

$$(1 - n_3^2)p^4 - (n_1^2 + n_2^2)p^2 + n_1^2 n_2^2 = 0. \qquad (a)'$$

It should be noted that the quantities n_1 and n_2 have simple physical meanings, thus, n_1 represents the frequency of oscillation of the upper body if the mass of the lower body is thought of as being concentrated at the hinge B. n_2 is the frequency of oscillation of the lower body if the hinge B is at rest. In discussing the frequency equation (i) it was pointed out that the left side of the equation is positive for $p^2 = 0$, and for $p^2 = \infty$, but it is negative for $p^2 = c_{11}/a_{11}$ and for $p^2 = c_{22}/a_{22}$. Hence the smaller root of the equation $(a)'$ must be smaller than n_1 and n_2, and the larger root must be larger than n_1 and n_2. The expressions for these roots are

$$p_1^2 = \frac{1}{2(1 - n_3^2)}\left(n_1^2 + n_2^2 - \sqrt{(n_2^2 - n_1^2)^2 + 4n_1^2 n_2^2 n_3^2}\right),$$

$$p_2^2 = \frac{1}{2(1 - n_3^2)}\left(n_1^2 + n_2^2 + \sqrt{(n_2^2 - n_1^2)^2 + 4n_1^2 n_2^2 n_3^2}\right).$$

The ratios of the amplitudes of the corresponding modes of vibration are, from eq. (j),

$$\left(\frac{\varphi_1}{\varphi_2}\right)_1 = \frac{a_{12}p_1^2 - c_{12}}{c_{11} - a_{11}p_1^2} = \frac{a_{12}}{a_{11}}\frac{p_1^2}{n_1^2 - p_1^2}, \qquad (b)'$$

$$\left(\frac{\varphi_1}{\varphi_2}\right)_2 = \frac{a_{12}p_2^2 - c_{12}}{c_{11} - a_{11}p_2^2} = \frac{a_{12}}{a_{11}}\frac{p_2^2}{n_1^2 - p_2^2}. \qquad (c)'$$

Assuming that $p_1 < p_2$ we find that for the mode with lower frequency the ratio of the amplitudes is positive and for the higher frequency it is negative. These two modes of

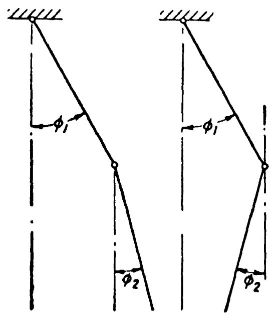

FIG. 120.

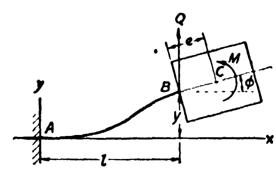

FIG. 121.

vibration are shown diagramatically in Fig. 120. Having found the principal modes of vibration we obtain the general solution by superposing the two modes of vibration with proper amplitudes and with proper phase angles so as to satisfy the initial conditions. If the system is to vibrate in one of its principal modes the ratio between the angles φ_1 and φ_2, given by eq. $(b)'$ or eq. $(c)'$, must be established initially before the system is relieved without initial velocities.

3. Investigate the small vibration in the horizontal plane xy of a plate BC, Fig. 121, attached to a prismatical bar AB. Assume that the xy plane is a principal plane of the bar, and that the center of gravity of the plate C is on the prolongation of the axis of the bar; moreover, let us neglect the mass of the bar.*

Solution. The position of the plate in xy plane is completely defined by the deflection y of the end B of the bar and by the angle φ of the tangent to the deflection curve. We take these two quantities as generalized coordinates of the moving plate. The positive directions of these coordinates are indicated in the figure. The corresponding generalized forces are the transverse force Q and the couple M. The directions of the force and of the couple shown in the figure are the positive directions when we are considering the action on the plate, but when we are dealing with the action on the bar the directions must be reversed. From the elementary formulae of strength of material and by noting the above mentioned agreement in regard to signs we have the following expressions for the deflection y and for the angle φ:

$$
\left.\begin{aligned}
y &= -\left(\frac{Ql^3}{3EI} + \frac{Ml^2}{2EI}\right) \\
\varphi &= -\left(\frac{Ql^2}{2EI} + \frac{Ml}{EI}\right)
\end{aligned}\right\} \qquad (d)'
$$

in which EI is flexural rigidity of the bar in the xy plane. The kinetic energy of the system consists of energy of rotation of the plate about its center of gravity C, and of translatory energy of the plate center. Thus

$$
T = \frac{mi^2\dot{\varphi}^2}{2} + \frac{m}{2}(\dot{y} + e\dot{\varphi})^2, \qquad (e)'
$$

where i is the radius of gyration of the plate with respect to the axis normal to the plate through C and e is the distance BC. Substituting T in Lagrange's equations (72) we obtain

$$
\begin{aligned}
m(\ddot{y} + e\ddot{\varphi}) &= Q, \\
m[e\ddot{y} + (e^2 + i^2)\ddot{\varphi}] &= M,
\end{aligned}
$$

with these expressions for the generalized forces Q and M the equations $(d)'$ become

$$
\begin{aligned}
y &= -\frac{l^3}{3EI}m(\ddot{y} + e\ddot{\varphi}) - \frac{l^2}{2EI}m[e\ddot{y} + (e^2 + i^2)\ddot{\varphi}], \\
\varphi &= -\frac{l^2}{2EI}m(\ddot{y} + e\ddot{\varphi}) - \frac{l}{EI}m[e\ddot{y} + (e^2 + i^2)\ddot{\varphi}].
\end{aligned}
$$

Taking the solution of these equations in the form

$$
y = \lambda_1 \cos(pt + \alpha), \quad \varphi = \lambda_2 \cos(pt + \alpha),
$$

and proceeding as before we obtain a quadratic frequency equation for p^2 the roots of which are

$$
p_{1,2}^2 = \frac{6EI}{ml^3} \frac{1}{1 + \frac{3e}{l} + \frac{3(e^2 + i^2)}{l^2} \pm \sqrt{\left[1 + \frac{3e}{l} + \frac{3(e^2 + i^2)}{l^2}\right]^2 - 3\frac{i^2}{l^2}}}. \qquad (f)
$$

* See M. Rössiger, Annalen d. Physic, 5 series, v. 15, p. 735, 1932.

In a particular case when the mass is concentrated at the end of the cantilever we have $e = i = 0$ and $(f)'$ reduces to

$$p_1^2 = \frac{3EI}{ml^3}, \quad p_2^2 = \infty.$$

The first of these solutions can be easily obtained by considering the system in Fig. 121 a one degree of freedom system and by neglecting the rotatory inertia of the plate at the end. The second of these solutions states that if the rotatory inertia approaches zero the corresponding frequency becomes infinitely large.

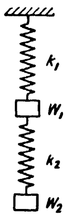

FIG. 122.

4. Determine the two natural frequencies of the vertical vibrations of the system shown in Fig. 122, if the weights W_1 and W_2 are 20 lb. and 10 lb. respectively; and if the spring constants k_1 and k_2 are 200 and 100 lb. per inch. Find the ratio a_1/a_2 of the amplitudes of W_1 and W_2 for the two principal modes of vibration.

Solution. The squares of the circular frequencies are $p_1^2 = 1930$ and $p_2^2 = 7720$. The corresponding ratios of the amplitudes are $a_1/a_2 = \frac{1}{2}$ and $a_1/a_2 = -1$.

35. Particular Cases.—In the previous discussion vibrations about a position of stable equilibrium of a system were considered. The expressions for the potential energy were always positive and conditions given by (d) (see p. 195) were satisfied. Let us now consider a particular case when the last of the three requirements (d) is not fulfilled, moreover let us assume that

$$c_{11}c_{22} - c_{12}^2 = 0. \tag{a}$$

In such a case it is possible to have displacements that do not produce any change in the potential energy of the system;* thus the system is in a position of indifferent equilibrium with respect to such displacements. It is also seen that the frequency equation (i) (see p. 196) has a root $p^2 = 0$. In discussing the physical significance of this solution, let us consider an example shown in Fig. 123. The shaft with two discs at the ends represents a system with two degrees of freedom so that two coordinates, say two angles of rotation φ_1 and φ_2, are needed to specify the configuration of the system. The potential energy of the system depends only on the angle of twist of the shaft, equal to $\varphi_2 - \varphi_1$, and a rotation of the system as a rigid body does not contribute to the potential energy; thus we have the particular case discussed above. Using the notations:

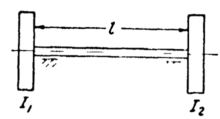

FIG. 123.

I_1 and I_2 – moments of inertia of the discs,
J_p – polar moment of inertia of the shaft,
G – modulus of elasticity in shear,

the expressions for potential and kinetic energy become

$$V = \frac{GJ_p}{2l}(\varphi_2 - \varphi_1)^2,$$
$$T = \frac{1}{2}(\dot{\varphi}_1^2 I_1 + \dot{\varphi}_2^2 I_2). \tag{b}$$

* It is only necessary to have $c_{12}q_1 + c_{22}q_2 = 0$ in expression $(b)'$ (see page. 195).

Comparing the expression for V with expression (b) of the previous article, p. 195, it is seen that in the case under consideration $c_{11} = c_{22} = -c_{12}$. Thus condition ($a$) is satisfied and one of the roots of the frequency equation will be equal to zero.

In our further discussion we introduce as one of the coordinates the angle of twist ψ and as the second coordinate, the angle of rotation φ_2. Then $\varphi_1 = \varphi_2 - \psi$, and our expressions ($b$) become

$$V = \frac{GJ_p}{2l}\psi^2,$$

$$T = \frac{I_1}{2}(\dot{\varphi}_2^2 - 2\dot{\varphi}_2\dot{\psi} + \dot{\psi}^2) + \frac{I_2}{2}\dot{\varphi}_2^2.$$

Substituting in Lagrange's equations, we obtain

$$(I_1 + I_2)\ddot{\varphi}_2 - I_1\ddot{\psi} = 0,$$

$$I_1(\ddot{\psi} - \ddot{\varphi}_2) + \frac{GJ_p}{l}\psi = 0. \qquad (c)$$

Eliminating φ_2, we find that

$$\frac{I_1 I_2}{I_1 + I_2}\ddot{\psi} + \frac{GJ_p}{l}\psi = 0.$$

From this equation we see that the frequency of torsional vibration p is identical with the one given by formula 17 (p. 12) and that the angle of twist can be represented by the following formula

$$\psi = a \sin (pt + \alpha),$$

in which the amplitude a and the phase angle α are to be determined from initial conditions. Substituting ψ in the first equations (c) we find

$$\varphi_2 = \frac{I_1}{I_1 + I_2} a \sin (pt + \alpha) + C_1 t + C_2.$$

It is seen that the coordinate ψ, relating to the stable equilibrium position of the system is varying during the motion within the limits $\pm a$, while the coordinate φ_2 relating to the indifferent equilibrium position of the system may grow indefinitely with time. Thus the motion of the system consists of a simple torsional vibration on which a uniform rotation with a constant angular velocity C_1 is superposed. Analogous conclusions will always be obtained when one of the roots of the frequency equation is zero.

As a second particular case let us consider problems in which the frequency equation (i) of the previous article has two equal roots. It was shown (see p. 196) that if we plot the values of the left side of eq. (i) against p^2 a curve is obtained which has negative ordinates for $p^2 = c_{11}/a_{11}$ as well as for $p^2 = c_{22}/a_{22}$ and that there are two intersection points with the abscissa axis that define the two different roots of the equation. However, in the particular case, when

$$\frac{c_{11}}{a_{11}} = \frac{c_{22}}{a_{22}} = \frac{c_{12}}{a_{12}},$$

the two intersection points coincide and we have two equal roots

$$p^2 = \frac{c_{11}}{a_{11}} = \frac{c_{22}}{a_{22}} = \frac{c_{12}}{a_{12}}. \qquad (d)$$

The expressions (*b*) and (*c*) of the previous article for the potential and for the kinetic energy can then be written as follows:

$$V = \tfrac{1}{2}p^2(a_{11}q_1^2 + 2a_{12}q_1q_2 + a_{22}q_2^2),$$
$$T = \tfrac{1}{2}(a_{11}\dot{q}_1^2 + 2a_{12}\dot{q}_1\dot{q}_2 + a_{22}\dot{q}_2^2).$$

Substituting these expressions in Lagrange's eq. 73, we obtain

$$a_{11}(\ddot{q}_1 + p^2q_1) + a_{12}(\ddot{q}_2 + p^2q_2) = 0,$$
$$a_{12}(\ddot{q}_1 + p^2q_1) + a_{22}(\ddot{q}_2 + p^2q_2) = 0,$$

and since $a_{11}a_{22} - a_{12}^2 \neq 0$, we must have

$$\ddot{q}_1 + p^2q_1 = 0,$$
$$\ddot{q}_2 + p^2q_2 = 0.$$

From these equations we conclude that

$$q_1 = a_1 \sin (pt + \alpha_1),$$
$$q_2 = a_2 \sin (pt + \alpha_2).$$

Thus in the case of equal roots both coordinates are represented by harmonic vibrations of the same frequency. The amplitudes and phase angles of these vibrations should be determined from initial conditions. As an example of such a system we have the case represented in Fig. 124.* Two equal masses m, joined by a horizontal bar AB, are suspended on two springs of equal rigidity having spring constants k. It is required to investigate the small vertical vibrations of the masses m, neglecting the mass of the bar. The position of the system can be completely defined by the vertical displacement y of the mid-point C and by the angle of rotation φ. The displacements of the masses in such cases are

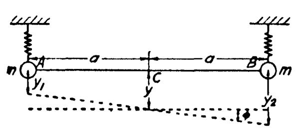

FIG. 124.

$$y_1 = y - a\varphi \quad \text{and} \quad y_2 = y + a\varphi,$$

and we obtain for the potential and for the kinetic energy of the system, the following expressions

$$V = k(y^2 + a^2\varphi^2),$$
$$T = m(\dot{y}^2 + a^2\dot{\varphi}^2).$$

It is seen that conditions (*d*) are satisfied and we have a system with two equal frequencies for the two modes of vibration.

36. Forced Vibrations.—In those cases where periodical disturbing forces are acting on the system forced vibrations will take place. By using Lagrange's equations in their general form (74) and substituting for T and V their general expressions (76) and (75) the equations of motion will be,

* A more general case is discussed in Art. 40.

$$
\begin{aligned}
&a_{11}\ddot{q}_1 + a_{12}\ddot{q}_2 + a_{13}\ddot{q}_3 + \cdots + c_{11}q_1 + c_{12}q_2 + c_{13}q_3 + \cdots = Q_1,\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&a_{n1}\ddot{q}_1 + a_{n2}\ddot{q}_2 + a_{n3}\ddot{q}_3 + \cdots + c_{n1}q_1 + c_{n2}q_2 + c_{n3}q_3 + \cdots = Q_n.
\end{aligned} \tag{a}
$$

We proceed to consider now the most important case where the generalized forces are of the simple harmonic type having the same period and the same phase so that every one of these forces can be represented in the form $Q_s = b_s \cos(\omega t + \beta)$, ω and β being constant.

A particular solution of eqs. (a) can be taken in the form:

$$
q_1 = \lambda_1 \cos(\omega t + \beta); \qquad q_2 = \lambda_2 \cos(\omega t + \beta); \cdots q_n = \lambda_n \cos(\omega t + \beta).
$$

Substituting in eqs. (a) we obtain

$$
\begin{aligned}
&(c_{11} - a_{11}\omega^2)\lambda_1 + (c_{12} - a_{12}\omega^2)\lambda_2 + \cdots + (c_{1n} - a_{1n}\omega^2)\lambda_n = b_1,\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&(c_{n1} - a_{n1}\omega^2)\lambda_1 + (c_{n2} - a_{n2}\omega^2)\lambda_2 + \cdots + (c_{nn} - a_{nn}\omega^2)\lambda_n = b_n.
\end{aligned} \tag{b}
$$

From these equations the amplitudes $\lambda_1, \lambda_2, \cdots \lambda_n$ of the vibrations can be found.

It should be noted that the left sides of eqs. (b) are of the same form as in eqs. (n) of Art. 34 and it is seen that when the determinant of the eqs. (b) approaches zero, i.e., when the period of the disturbing force approaches one of the natural periods of vibration, the amplitudes of vibration become very great. This is the phenomenon of *resonance* which was discussed before for the case of systems with one degree of freedom.

If the generalized coordinates $q_1, q_2, \cdots q_n$ are *normal* or *principal coordinates* of the system, the expressions for the kinetic and potential energies become

$$
\begin{aligned}
2T &= a_{11}\dot{q}_1^2 + a_{22}\dot{q}_2^2 + \cdots + a_{nn}\dot{q}_n^2,\\
2V &= c_{11}q_1^2 + c_{22}q_2^2 + \cdots + c_{nn}q_n^2.
\end{aligned} \tag{c}
$$

Substituting in Lagrange's equation (74) we obtain

$$
\begin{aligned}
&a_{11}\ddot{q}_1 + c_{11}q_1 = Q_1,\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&a_{nn}\ddot{q}_n + c_{nn}q_n = Q_n.
\end{aligned} \tag{78}
$$

These differential equations, each containing one coordinate only, are of the same kind as we had in the case of systems with one degree of freedom. Thus there is no difficulty in obtaining a general solution of these equations for any kind of disturbing forces. Assuming as before,

$$
\begin{aligned}
Q_s &= b_s \cos(\omega t + \beta), \\
q_1 &= \lambda_1 \cos(\omega t + \beta), \\
&\cdots\cdots\cdots \\
q_n &= \lambda_n \cos(\omega t + \beta),
\end{aligned} \tag{d}
$$

we have, from eqs. (78),

$$
\lambda_s = \frac{b_s}{c_{ss} - \omega^2 a_{ss}} = \frac{b_s}{c_{ss}} \frac{1}{1 - \dfrac{\omega^2}{p^2}}. \tag{e}
$$

Here b_s/c_{ss} represents the statical deflection produced by the force Q_s at the point of its application and ω^2/p^2 the square of the ratio between the frequency of the force and the frequency of natural vibration. An analogous result has been previously obtained for systems with one degree of freedom (see eq. 20) and it can be concluded that if a simple harmonic force corresponding to one of the principal coordinates of a system be assumed, the maximum displacement may be obtained by multiplying the static deflection by the magnification factor. The magnification factor has the same form as in the case of systems with one degree of freedom.

As an example of an application of the general theory of forced vibration, let us consider the vibration of a uniformly rotating disc on a flexible shaft AB, taking also into account the lateral flexibility of the columns supporting the bed plate, Fig. 125. We assume that the middle plane xy of the disc is the plane of symmetry of the structure and consider the motion of the disc in this plane. Let the origin of the coordinates, Fig. 125, c, coincide with the unstrained position of the axis of the shaft.* Moreover, let $\overline{OD} = \xi$ denote the horizontal displacements of the bed plate due to bending of the columns.† $\overline{DE}$ denotes the deflection of the shaft during vibration and E is the intersection point of the deflected axis of the shaft with the xy plane. $\overline{EC} = e$ is the small eccentricity, and C is the center of gravity of the disc. The position of the disc in the xy plane is completely defined by the coordinates x and y of the center of gravity C and by the angle of rotation φ. The position of the bed plate is defined by the horizontal deflection ξ. Denoting by m and by m_1 the masses of the bed plate and of the disc respectively and by I the moment of inertia of the disc about the axis of the shaft, we may write an expression for the kinetic energy of the vibrating system as follows:

$$
T = \frac{1}{2} m\dot{\xi}^2 + \frac{1}{2} m_1(\dot{x}^2 + \dot{y}^2) + I \frac{\dot{\varphi}^2}{2}. \tag{f}
$$

* The effect of a gravity force is neglected in this discussion. This effect is considered in another Article.

† Compression of columns is neglected in this discussion.

In calculating the potential energy of the system we denote by k the spring constant corresponding to the deflection ξ of the bed plate, and by k_1 the spring constant relating to the deflection f of the shaft. Then

$$V = \frac{k}{2}\xi^2 + \frac{k_1}{2}f^2.$$

This expression may be written in a final form by considering the geometry of Fig. 125, *c*:

$$f^2 = \overline{DE}^2 = (x - \xi - e\cos\varphi)^2 + (y - e\sin\varphi)^2.$$

Then

$$V = \frac{k}{2}\xi^2 + \frac{k_1}{2}[(x - \xi - e\cos\varphi)^2 + (y - e\sin\varphi)^2]. \tag{e}$$

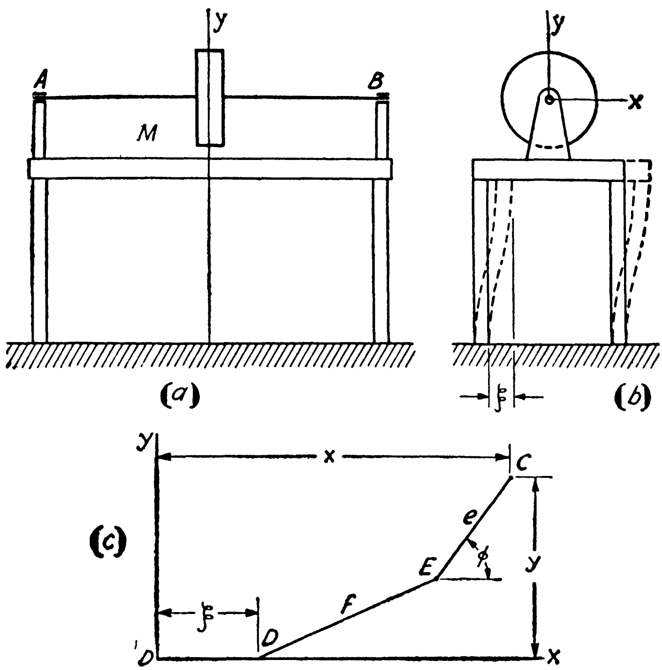

FIG. 125.

Substituting expressions (*f*) and (*e*) in Lagrange's equations and assuming that a torque M_t is the only generalized force acting on the system, the equations of motion become

$$\begin{aligned} m\ddot{\xi} + k\xi - k_1(x - \xi - e\cos\varphi) &= 0, \\ m_1\ddot{x} + k_1(x - \xi - e\cos\varphi) &= 0, \\ m_1\ddot{y} + k_1(y - e\sin\varphi) &= 0, \\ I\ddot{\varphi} + ek_1[(x - \xi - e\cos\varphi)\sin\varphi - (y - e\sin\varphi)\cos\varphi] &= M_t. \end{aligned} \tag{g}$$

It was tacitly assumed that the torque applied is such as to maintain a uniform rotation. Denoting the speed of this rotation by ω, we have $\varphi = \omega t$.

Substituting this into the first three of equations (g), we find

$$\begin{aligned} m\ddot{\xi} + (k + k_1)\xi - k_1 x &= - ek_1 \cos \omega t, \\ m_1\ddot{x} + k_1 x - k_1\xi &= ek_1 \cos \omega t, \\ m_1\ddot{y} + k_1 y &= k_1 e \sin \omega t. \end{aligned} \tag{h}$$

These are the equations of the forced vibrations of the system. It is seen that the third equation contains only the coordinate y. Thus the vertical vibrations of the shaft are not effected by the flexibility of the columns, and the corresponding critical speed is

$$\omega_1 = \sqrt{\frac{k_1}{m_1}}. \tag{i}$$

In other words it is the same as for a shaft in rigid bearings. The first two of equations (h) give us the horizontal vibrations of the disc and of the bed plate. We take the solutions of these equations in the form

$$x = \lambda_1 \cos \omega t \qquad \xi = \lambda_2 \cos \omega t.$$

Substituting in the equations, we obtain

$$\begin{aligned} (-m_1\omega^2 + k_1)\lambda_1 - k_1\lambda_2 &= ek_1, \\ -k_1\lambda_1 + (-m\omega^2 + k + k_1)\lambda_2 &= -ek_1, \end{aligned} \tag{j}$$

from which the amplitudes λ_1 and λ_2 can be calculated. The corresponding critical speeds are obtained by equating the determinant of these equations to zero. Thus we find

$$(-m_1\omega^2 + k_1)(-m\omega^2 + k + k_1) - k_1^2 = 0,$$

or

$$(-m_1\omega^2 + k_1)(-m\omega^2 + k) - k_1 m_1 \omega^2 = 0. \tag{k}$$

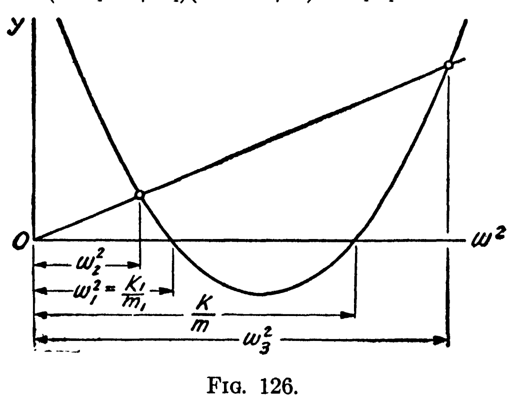

Fig. 126.

Taking ω^2 as abscissas and the magnitudes of the first term on the left side of eq. (k) as ordinates a parabola is obtained (Fig. 126) intersecting the horizontal axis at $\omega^2 = k_1/m_1$ and $\omega^2 = k/m$. The critical speeds ω_2 and ω_3 are determined by the intersection points of the parabola with the inclined straight line $y = k_1 m_1 \omega^2$ as shown in the figure. It is seen that one of these speeds is less and the other is larger than the critical speed (i) for the vertical vibrations.

If the angular velocity ω is different from the above determined critical values, the determinant of equations (j), represented by the left side of (k), is different from zero. Denoting its value by Δ, we find from (j)

$$\lambda_1 = \frac{ek_1(-m\omega^2 + k + k_1) - ek_1^2}{\Delta}, \quad \lambda_2 = \frac{-ek_1(-m_1\omega^2 + k_1) + ek_1^2}{\Delta},$$

which determine the amplitudes of the horizontal forced vibrations.*

37. Vibration with Viscous Damping.—In a general discussion of the effect of damping on vibrations it is advantageous to introduce the notion of the rate at which energy is dissipated. Considering first a particle moving rectilinearly along an x-axis, we may take the resisting force of a viscous damping equal to $f = -c\dot{x}$, where the minus sign indicates that the force acts in the direction opposite to the velocity, and the constant coefficient c is the magnitude of the friction force when the velocity is unity. The work done by the friction force during a small displacement δx is then $-c\dot{x}\delta x$ and the amount of energy dissipated is

$$c\dot{x}\delta x = c\dot{x}^2\delta t$$

so that the time rate at which energy is dissipated in this case is $c\dot{x}^2$. In the further discussion we introduce the dissipation function F which represents half the rate at which energy is dissipated. Then

$$F = \tfrac{1}{2} c\dot{x}^2 \tag{a}$$

and the friction force can be obtained by differentiation;

$$f = -c\dot{x} = -\frac{dF}{d\dot{x}}. \tag{b}$$

In the general case of motion of a particle the velocity can be resolved into three orthogonal components so that the dissipation function becomes

$$F = \tfrac{1}{2}(c_1\dot{x}^2 + c_2\dot{y}^2 + c_3\dot{z}^2). \tag{c}$$

The factors c_1, c_2, c_3 being the constants defining the viscous friction in the x, y and z directions.

In the case of a system of particles the dissipation function can be obtained by a summation of expressions (c) for all particles involved.

$$F = \tfrac{1}{2}\sum(c_1\dot{x}^2 + c_2\dot{y}^2 + c_3\dot{z}^2). \tag{d}$$

* Vibration of rotors in flexible bearings has been discussed by V. Blaess, Maschinenbau-Betrieb, 1923, p. 281. See also D. M. Smith, Proc. Roy. Soc. A, V. 142, p. 22, 1933.

If x, y and z be expressed by the generalized coordinates (see eqs. (a), p. 189) the dissipation function can be represented as a function of the second degree of the generalized velocities $\dot{q}_1, \dot{q}_2 \cdots$ and we obtain *

$$F = \tfrac{1}{2} b_{11}\dot{q}_1^2 + b_{12}\dot{q}_1\dot{q}_2 + \tfrac{1}{2} b_{22}\dot{q}_2^2 + \cdots. \qquad (e)$$

Here the coefficients $b_{11}, b_{12}, \cdots$ generally depend on the configuration of the system. But in the case of small vibrations in the neighborhood of a configuration of stable equilibrium these coefficients can be treated as being constants. The friction force f_i corresponding to any generalized coordinate q_1 may now be obtained by differentiation of expression (e)

$$f_i = -\frac{\partial F}{\partial \dot{q}_i}.$$

Introducing this expression into the Lagrangian eqs. (74) we obtain the following equations that will take care of viscous friction.

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} + \frac{\partial F}{\partial \dot{q}_i} = Q_i. \qquad (79)$$

Let us apply these equations to systems with two degrees of freedom vibrating in the neighborhood of a configuration of stable equilibrium and in doing so let us assume that the coordinates q_1 and q_2 are the principal coordinates of the system. Then the expression for the kinetic energy contains only terms with squares of the velocities $\dot{q}_1, \dot{q}_2$ and the expression for the potential energy contains only the squares of the coordinates q_1, q_2 so that we have

$$\begin{aligned} T &= \tfrac{1}{2}(a_{11}\dot{q}_1^2 + a_{22}\dot{q}_2^2) \\ V &= \tfrac{1}{2}(c_{11}q_1^2 + c_{22}q_2^2) \\ F &= \tfrac{1}{2}(b_{11}\dot{q}_1^2 + 2b_{12}\dot{q}_1\dot{q}_2 + b_{22}\dot{q}_2^2). \end{aligned} \qquad (f)$$

From the fact that the kinetic as well as the potential energy is always positive, it follows that:

$$a_{11} > 0, \qquad a_{22} > 0, \qquad c_{11} > 0, \qquad c_{22} > 0. \qquad (g)$$

Regarding the dissipation function F it can also be stated that it must always be positive since we have friction forces resisting the motion whatever be the possible displacement. Hence (see p. 195)

$$b_{11} > 0, \qquad b_{22} > 0, \qquad b_{11}b_{22} - b_{12}^2 > 0. \qquad (h)$$

* The Dissipation Function was introduced for the first time by Lord Rayleigh, Proc. of the Mathematical Society, 1873. See also his Theory of Sound, 2nd ed. v. 1, p. 103.

Substituting expressions (f) into Lagrange's equation and considering only the free vibrations of the system, i.e., $Q_1 = Q_2 = 0$, we obtain the following equations of motion

$$\begin{aligned} a_{11}\ddot{q}_1 + b_{11}\dot{q}_1 + b_{12}\dot{q}_2 + c_{11}q_1 &= 0 \\ a_{22}\ddot{q}_2 + b_{22}\dot{q}_2 + b_{12}\dot{q}_1 + c_{22}q_2 &= 0. \end{aligned} \tag{i}$$

Thus we have a system of linear equations with constant coefficients. The general method of solving these equations is to assume a solution in the following form for q_1 and q_2

$$q_1 = C_1e^{st}, \qquad q_2 = C_2e^{st}. \tag{j}$$

Substituting these expressions in eqs. (i) we find the following equations for determining C_1, C_2 and s

$$\begin{aligned} C_1(a_{11}s^2 + b_{11}s + c_{11}) + C_2b_{12}s &= 0 \\ C_1b_{12}s + C_2(a_{22}s^2 + b_{22}s + c_{22}) &= 0. \end{aligned} \tag{k}$$

These two linear, homogeneous equations may give for C_1 and C_2 solutions different from zero only if their determinant is zero. Equating this determinant to zero we obtain the following equation for calculating s

$$(a_{11}s^2 + b_{11}s + c_{11})(a_{22}s^2 + b_{22}s + c_{22}) - b_{12}{}^2s^2 = 0. \tag{l)$'$}$$

This is an equation of the fourth degree in s and we shall have four roots which give four particular solutions of eqs. (i) when substituted in (j). By combining these four solutions, the general solution of eqs. (i) is obtained.

If conditions (g) and (h) are satisfied, all four roots of eq. (l) are complex with negative real parts * and we shall have

$$\begin{aligned} s_1 &= -n_1 + ip_1 \\ s_2 &= -n_1 - ip_1 \\ s_3 &= -n_2 + ip_2 \\ s_4 &= -n_2 - ip_2 \end{aligned} \tag{l}$$

where n_1 and n_2 are positive numbers. Substituting each of these roots in eqs. (k) the ratios such as C_1/C_2 for each root will be obtained. Thus we find four particular solutions of the type (j) with four constants of integration which can be determined from four initial conditions, namely from the initial values of the coordinates q_1, q_2 and their derivatives $\dot{q}_1$ and $\dot{q}_2$.

* The general proof of this statement was given by A. Hurwitz, Math. Ann. v. 46, p. 273, 1895. The proof can be found in Riemann-Webers "Differentialgleichungen der Physik," v. 1, p. 125, 1925

It is advantageous to proceed as in the case of systems with one degree of freedom (see Art. 8) and introduce trigonometric functions instead of exponential functions (j). Taking the first two roots $(l)'$ and observing that

$$
\begin{aligned}
e^{(-n_1 + ip_1)t} + e^{(-n_1 - ip_1)t} &= 2e^{-n_1t} \cos p_1t \\
e^{(-n_1 + ip_1)t} - e^{(-n_1 - ip_1)t} &= 2ie^{-n_1t} \sin p_1t
\end{aligned}
$$

we can represent the combination of the first two particular solutions (j) in the following form

$$
\begin{aligned}
q_1 &= e^{-n_1t}\cdot(C_1' \cos p_1t + C_2' \sin p_1t) \\
q_2 &= e^{-n_1t} (C_1'' \cos p_1t + C_2'' \sin p_1t).
\end{aligned}
$$

Thus each coordinate represents a vibration with damping similar to what we had in the case of systems with one degree of freedom. The real part n_1 of the roots defines the rate at which the amplitudes of vibration are damped out and the imaginary part p_1 defines the frequency of vibrations.

In the same manner the last two roots $(l)'$ can be treated and finally we obtain the general solution of eqs. (i) in the following form

$$
\begin{aligned}
q_1 &= e^{-n_1t}(C_1' \cos p_1t + C_2' \sin p_1t) + e^{-n_2t}(D_1' \cos p_2t + D_2' \sin p_2t) \\
q_2 &= e^{-n_1t}(C_1'' \cos p_1t + C_2'' \sin p_1t) + e^{-n_2t}(D_1'' \cos p_2t + D_2'' \sin p_2t).
\end{aligned}
\tag{m}
$$

Owing to the fact that the ratio between the constants C_1, C_2 is determined from eqs. (k) for each particular solution (j), there will be only four independent constants in expressions (m) to be determined from the initial conditions of the system.

In the case of small damping the numbers n_1 and n_2 in roots (l) are small and the effects of damping on the frequencies of vibrations are negligibly small quantities of the second order. Thus the frequencies p_1 and p_2 can be taken equal to the frequencies of vibrations without damping.

If we have a system with very large damping it is possible that two or all four roots $(l)'$ become real and negative. Assuming, for instance, that the last two roots are real, we shall find, as in the case of systems with one degree of freedom (p. 37), that the corresponding motion is aperiodic and that the complete expression for the motion will consist of damped vibrations superposed on aperiodic motion.

Some examples of vibrations with damping are discussed in Art. 41.

38. Stability of Motion.—In our previous discussion we had several examples of instability of motion. One example of this kind occurred when we considered a vertically hanging pendulum of which the point of suspension oscillated vertically. We have found (see p. 158) that at a certain frequency of these oscillations the vertical position of the pendulum becomes unstable and lateral vibrations are being built up gradually. Another example of the same kind we had in the case of a rotating shaft (p. 159). Sometimes it is desirable to investigate a certain steady motion of a system and to decide if this motion is stable or unstable. The general method used in such cases is: (1) to assume that a small deviation or displacement from the steady form of motion is produced, (2) to investigate the resulting vibrations of the system with respect to the steady motion caused by the small deviation or displacement; (3) if these vibrations, as in the case of vibrations with viscous damping of the previous article, have the tendency to die out, we conclude that the steady motion is stable. Otherwise this motion is unstable. Thus the question of stability of motion requires an investigation of the small vibrations with respect to the steady motion of the system resulting from arbitrarily assumed deviations or displacements from the steady form of motion. Mathematically, such an investigation results in a system of linear differential equations similar to eqs. (i) of the previous article, and the question of stability or of instability of the steady motion depends on the roots of an algebraic equation similar to eq. (l) (p. 215). If all the roots have negative real parts, as was the case in the previous article, the vibration caused by the arbitrary deviation will be damped out, which means that the steady motion under consideration is stable. Otherwise the steady motion will be unstable.

Certain requirements regarding the coefficients of the algebraic equation, resulting from the differential equations similar to eqs. (i), have been established so that we can decide about the sign of real parts of the roots without solving the equations.* If we have, for instance, a cubic equation:

$$a_0s^3 + a_1s^2 + a_2s + a_3 = 0,$$

all the roots will have a negative real part and, consequently, the motion will be stable if all the coefficients of the equation are positive and if

$$a_1a_2 - a_0a_3 > 0. \tag{a}$$

In the case of an equation of the fourth degree

$$a_0s^4 + a_1s^3 + a_2s^2 + a_3s + a_4 = 0,$$

* Such rules were established by E. J. Routh, "On the Stability of a Given Motion," London, 1877; see also his "Rigid Dynamics," vol. 2 and the paper by A. Hurwitz, loc. cit., p. 215.

for stability of motion it is again necessary to have all the coefficients positive and also that:

$$a_3(a_1a_2 - a_0a_3) - a_1^2a_4 > 0. \tag{b}$$

Let us apply this general consideration of stability problems to particular cases. As a first example we will consider the stability of rotation of a pendulum with respect to its vertical axis 0–0, Fig. 127. The experiments show that if the angular velocity of rotation ω is below a certain limiting value, the rotation is stable and if by an arbitrary lateral impulse lateral oscillations of the pendulum about the horizontal pin A are produced, these oscillations gradually die out. If the angular velocity ω is above the limiting value, the vertical position of the pendulum is unstable and the slightest lateral force will produce a large deflection of the pendulum from its vertical position. In our discussion let us assume that the angular velocity of rotation about the vertical axis is constant and that the mass m of the pendulum can be assumed concentrated at the center C of the bob. If a lateral motion of the pendulum, defined by a small angle α, takes place, the velocity of the center C consists of two components: (1) a velocity of lateral motion $l\dot{\alpha}$, and (2) a velocity of rotation about the axis 0–0, equal to $\omega l \sin \alpha \approx \omega l \alpha$. The kinetic energy of the system is then

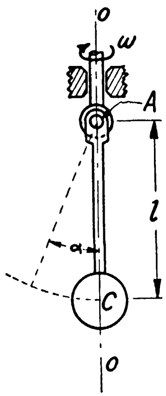

FIG. 127.

$$T = \frac{ml^2\dot{\alpha}^2}{2} + \frac{m\omega^2l^2\alpha^2}{2}.$$

The potential energy of the system, due to the gravity force, is

$$V = mgl(1 - \cos \alpha) \approx \frac{mgl\alpha^2}{2}.$$

Substituting V and T in Lagrange's equation we obtain

$$ml^2\ddot{\alpha} - m\omega^2l^2\alpha + mgl\alpha = 0$$

or

$$\ddot{\alpha} + \left(\frac{g}{l} - \omega^2\right)\alpha = 0. \tag{c}$$

If

$$\frac{g}{l} - \omega^2 > 0, \tag{d}$$

eq. (c) defines a simple harmonic oscillation, which, due to unavoidable friction, will gradually die out. Thus the steady rotation of the pendulum in this case is stable. If

$$\frac{g}{l} - \omega^2 < 0, \tag{e}$$

eq. (c) will have the same form as for an inverted pendulum so that, instead of oscillating, the angle α will grow continuously. Thus the rotation of the pendulum in this case is unstable. The limiting value of the angular velocity is

$$\omega = \sqrt{\frac{g}{l}}. \tag{f}$$

In other words, the limiting angular speed is that speed at which the number of revolutions per second of the pendulum about the vertical axis is equal to the frequency of its free lateral oscillations.

If we assume that there is viscous friction in the pendulum we shall have the following equation instead of eq. (c):

$$\ddot{\alpha} + 2n\dot{\alpha} + \left(\frac{g}{l} - \omega^2\right)\alpha = 0. \tag{g}$$

If condition (d) is fulfilled, we obtain damped vibrations. If condition (e) exists, we can put eq. (g) into the following form:

$$\ddot{\alpha} + 2n\dot{\alpha} - p^2\alpha = 0,$$

where $p^2 = \omega^2 - \dfrac{g}{l}$.

Taking the solution of this equation in the form $\alpha = e^{st}$, we find that

$$s^2 + 2ns - p^2 = 0,$$

from which

$$s = -n \pm \sqrt{n^2 + p^2}.$$

It is seen that one of the roots is positive. Thus the angle α has a tendency to grow and the rotation is unstable.

Vibration of a Steam Engine Governor.—As a second example let us consider the stability of a steady rotation of a steam engine governor, shown in Fig. 128. Due to the centrifugal forces of the flyballs a compression of the governor's spring is produced by the sleeve B which is in direct mechanical connection with the steam supply throttle valve. If, for some reason, the speed of the engine increases, the rotational speed of the governor, directly connected to the engine's shaft, increases also. The flyballs then rise higher and thereby lift the sleeve so that the opening of the steam valve C is reduced which means that the engine is throttled down. On the other hand, if the engine speed decreases below normal, the flyballs move downward and thereby increase the opening of the valve and the amount of steam admitted to the engine. To simplify our discussion, let us assume that the masses of the flyballs are each equal to $m_2/2$ and the mass of the sleeve is m_1, moreover that all masses are concentrated at the centers of gravity and that the masses of the inclined bars and of the spring can be neglected. As coordinates of the system we take the angle of rotation φ of the governor about its vertical axis and the angle of inclination α which the bars of the governor are making with the vertical axis. The velocity of the centers of the flyballs consists of two components, (1) the

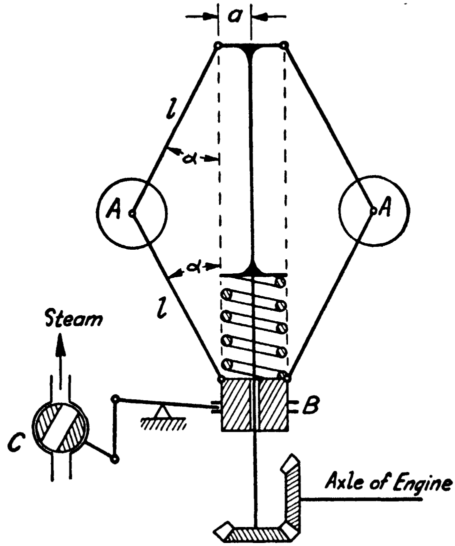

FIG. 128.

velocity of rotation $\dot{\varphi}(a + l \sin \alpha)$, and (2) the velocity of lateral motion $l\dot{\alpha}$. The vertical displacement of the sleeve from the lowest position when $\alpha = 0$ is $2l(1 - \cos \alpha)$, and the corresponding velocity is $2l\dot{\alpha} \sin \alpha$. The kinetic energy of the system is:

$$T = \tfrac{1}{2} m_2[(a + l \sin \alpha)^2\dot{\varphi}^2 + l^2\dot{\alpha}^2] + \frac{m_1}{2} 4l^2 \sin^2\alpha\dot{\alpha}^2 + \tfrac{1}{2} I\dot{\varphi}^2, \qquad (h)$$

where I is the reduced moment of intertia of the engine. The potential energy of the system consists, (1) of the energy due to gravity force

$$m_1g2l(1 - \cos \alpha) + m_2gl(1 - \cos \alpha),$$

and (2) of the strain energy of the spring.*

$$\frac{k4l^2(1 - \cos \alpha)^2}{2},$$

* It is assumed that for $\alpha = 0$ there is no stress in the spring.

where k is the spring constant. Thus the total potential energy is

$$V = gl(1 - \cos\alpha)(2m_1 + m_2) + 2kl^2(1 - \cos\alpha)^2. \tag{i}$$

We assume that there is a viscous damping opposing the vertical motion of the sleeve proportional to the sleeve velocity, $2l\dot{\alpha}\sin\alpha$. If the factor of proportionality be denoted by c, the rate at which energy is dissipated is $c(2l\dot{\alpha}\sin\alpha)^2$, and we obtain for the dissipation function the expression

$$F = \tfrac{1}{2}c(2l\dot{\alpha}\sin\alpha)^2. \tag{j}$$

Substituting expressions (h), (i) and (j) in Lagrange's equation (79), p. 214, we obtain the following two equations:

$$\begin{aligned}\ddot{\alpha}l^2(m_2 + 4m_1\sin^2\alpha) - m_2 l\cos\alpha(a + l\sin\alpha)\dot{\varphi}^2 - 4m_1l^2\sin\alpha\cos\alpha\dot{\alpha}^2 \\ = -gl\sin\alpha(2m_1 + m_2) - 4kl^2\sin\alpha(1 - \cos\alpha) - 4l^2c\dot{\alpha}\sin^2\alpha, \\ [I + m_2(a + l\sin\alpha)^2]\ddot{\varphi} = M,\end{aligned} \tag{k}$$

where M denotes the reduced torque acting on the engine shaft.

Let us consider first steady motion when $M = 0$. Then $\dot{\varphi} = \omega_0$, $\ddot{\varphi} = 0$, $\alpha = \alpha_0$, $\dot{\alpha} = 0$, $\ddot{\alpha} = 0$ and we obtain from the first equation

$$m_2 l\cos\alpha(a + l\sin\alpha)\omega^2 = gl\sin\alpha(2m_1 + m_2) + 4kl^2\sin\alpha(1 - \cos\alpha). \tag{l}$$

This equation can be readily deduced from statical consideration by applying fictitious centrifugal forces to the flyballs.

Let us now consider small vibrations about the steady motion discussed above. In such a case

$$\dot{\varphi} = \omega_0 + \omega \quad \text{and} \quad \alpha = \alpha_0 + \eta, \tag{m}$$

where ω denotes a small fluctuation in the angular velocity of rotation, and η a small fluctuation in the angle of inclination α. Substituting expressions (m) into equations (k) and keeping only small quantities of the first order we can put

$$\begin{gathered}\dot{\varphi}^2 = \omega_0^2 + 2\omega_0\omega, \quad \sin\alpha = \sin(\alpha_0 + \eta) = \sin\alpha_0 + \eta\cos\alpha_0 \\ \cos(\alpha_0 + \eta) = \cos\alpha_0 - \eta\sin\alpha_0.\end{gathered}$$

Then equations (k), with the use of eq. (l), become

$$\begin{aligned}m\ddot{\eta} + b\dot{\eta} + d\eta - e\omega = 0, \\ I_0\dot{\omega} = -f\eta,\end{aligned} \tag{n}$$

where

$$\begin{aligned}m &= l^2(m_2 + 4m_1\sin^2\alpha_0), \\ b &= 4cl^2\sin^2\alpha_0, \\ d &= m_2\omega_0^2[l\sin\alpha_0(a + l\sin\alpha_0) - l^2\cos^2\alpha_0] + gl\cos\alpha_0(2m_1 + m_2) \\ &\qquad + 4kl^2[\cos\alpha_0 - \cos^2\alpha_0 + \sin^2\alpha_0], \\ e &= 2\omega_0 l(a + l\sin\alpha_0)m_2, \\ I_0 &= I + m_2(a + l\sin\alpha_0)^2.\end{aligned}$$

f denotes the characteristic torque change factor of the engine, defined as $\dfrac{dM}{d\alpha}$ or as $\dfrac{\Delta M}{\eta}$, or, in other words, as the factor which, multiplied by the angular change η, gives the change in torque acting on the shaft of the engine. Thus the vibration of the governor with respect to the steady motion is defined by the system of linear equations (n). Assuming solutions of these equations in the form

$$\eta = C_1 e^{st}, \quad \omega = C_2 e^{st},$$

and substituting these expressions in (n), we obtain

$$\begin{aligned} C_1(ms^2 + bs + d) - eC_2 &= 0, \\ C_1 f + I_0 s C_2 &= 0. \end{aligned}$$

Equating the determinant of these equations to zero we find

$$I_0 s(ms^2 + bs + d) + ef = 0,$$

or

$$s^3 + \frac{b}{m}s^2 + \frac{d}{m}s + \frac{ef}{mI_0} = 0.$$

All constants entering into this equation are positive,* so that by using condition (a) (p. 217) we can state that the motion of the governor will be stable if

$$\frac{bd}{m^2} > \frac{ef}{mI_0}.$$

From this it follows that for a stable state of motion the quantity b, depending on viscous damping in the governor, must satisfy the condition

$$b > \frac{mef}{dI_0}.$$

If this condition is not satisfied, vibrations of the governor produced by a sudden change in load of the engine, will not be damped out gradually and the well-known phenomenon of *hunting* of a governor occurs.†

The method used above in discussing the stability of a governor has been applied successfully in several other problems of practical importance as, for instance, airplane-flutter,‡ automobile "shimmy",§ and axial oscillations of steam turbines.¶

39. Whirling of a Rotating Shaft Caused by Hysteresis.—In our previous discussion of instability of motion of a rotating disc (see p. 92) it

* We assume that for any increase in angular velocity the corresponding angle α, as defined by eq. (l), increases also. In such a case expression (d), containing negative terms, is positive.

† In the case when the engine is rigidly coupled to an electric generator an additional term proportional to φ will enter into the second of equations (k) so that instead of equations (n) we obtain two equations of the second order. The stability discussion requires then an investigation of the roots of an equation of the 4th degree. Such an investigation was made by M. Stone. Trans. A.I.E.E., 1933, p. 332.

‡ W. Birnbaum, Zeitschr. f. angew. Math. Mech. v. 4, p. 277, 1924.

§ G. Becker, H. Fromm and H. Maruhn, "Schwingungen in Automobillenkungen," Berlin, 1931.

¶ J. G. Baker, paper before A.S.M.E. meeting, December, 1934, New York.

was assumed that the material of the shaft is perfectly elastic and any kind of damping has been neglected. On the basis of this assumption two forms of whirling of the shaft due to some eccentricity have been discussed, namely, (1) below the critical speed ω_{cr} and, (2) above the critical speed. It was found that in both cases the plane containing the bent axis of the shaft rotates with the same speed as the shaft itself. Both these forms of motion are theoretically stable * so that if a small deviation from the circular path of the center of gravity of the disc is produced by impact, for example, the result is that small vibrations in a radial and in a tangential direction are superposed on the circular motion of the center of gravity. The existence of such motion can be demonstrated by the use of a suitable stroboscope.† In this way it can also be shown that due to unavoidable damping the vibrations gradually die out if the speed of the shaft is below ω_{cr}. However, if it is above ω_{cr} a peculiar phenomenon sometimes can be observed, namely, that the plane of the bent shaft rotates at the speed ω_{cr} while the shaft itself is rotating at a higher speed ω. Sometimes this motion has a steady character and the deflection of the shaft remains constant. At other times the deflection tends to grow with time up to the instant when the disc strikes the guard. To explain this phenomenon the imperfection in the elastic properties of the shaft must be considered.

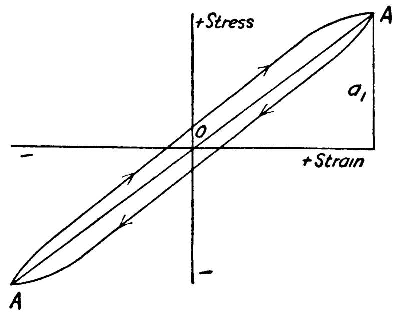

FIG. 129.

Experiments with tension-compression show that all materials exhibit some *hysteresis* characteristic so that instead of a straight line AA, Fig. 129, representing Hooke's law, we usually obtain a loop of which the width depends on the limiting values of stresses applied in the experiment. If the loading and unloading is repeated several hundred times, the shape of the loop is finally stabilized ‡ and the area of the loop gives the amount of energy dissipated per cycle due to hysteresis. We will now investigate

* The first investigation of this stability problem was made by A. Föppl, Der Civilingenieur, v. 41, p. 333, 1895.

† Experiments of this kind were recently made by D. Robertson, The Engineer, v. 156, p. 152, 1933, and v. 158, p. 216, 1934. See also his papers in Phil. Mag. ser. 7, v. 20, p. 793, 1935; and "The Institute of Mechanical Engrs.," October, 1935. In the last two papers a bibliography on the subject is given.

‡ We assume that the limits of loading are below the endurance limit of the material.

the effect of the hysteresis on bending of the shaft by first considering the case of static bending. We eliminate the effect of a gravity force by choosing a vertical shaft; moreover, we assume that it is deflected by a statically applied lateral force P in the plane of the figure (Fig. 130). The deflection δ may be taken proportional to the force

$$\delta = kP, \tag{a}$$

k being the spring constant of the shaft. In our further discussion we assume that the middle plane of the disc is the plane of symmetry of the shaft so that during bending the disc is moving parallel to itself. In Fig. 130b, the cross-section of the shaft is shown to a larger scale and the

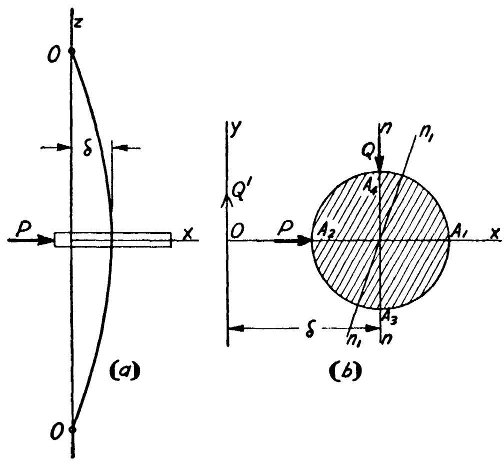

FIG. 130.

line n–n perpendicular to the plane of bending indicates the neutral line, so that the fibers of the shaft to the right of this line are in tension and to the left, in compression.

Let us now assume that a torque is applied in the plane of the disc so that the shaft is brought into rotation in a counter-clockwise direction, while the plane of bending of the shaft is stationary, i.e., the plane of the deflection curve of the axis of the shaft continues to remain in the xz plane. In this way the longitudinal fibers of the shaft will undergo reversal of stresses. For instance, a fiber A_1 at the convex side of the bent shaft is in tension, but after half a revolution of the shaft the fiber will be in compression at A_2 on the concave side. In the case of an ideal material, following Hooke's law, the relation between stress and strain is given by the straight line A–A in Fig. 129 and the distribution of bending stresses over the cross-section of the shaft will not be affected by the rotation. But the condition is different if the material exhibits hysteresis characteristics. From the loop in Fig. 129 we see that for the same strain we have two different values of stress corresponding to the upper (loading) and the lower (unloading) branch of the loop, respectively. Returning to the consideration of the cross-section of the rotating shaft in Fig. 130 *b*, we see that during the motion of the fiber from position A_2 to position A_1 the stress is varying from compression to tension, consequently we must use the upper branch of the loop. In the same way we conclude that during the motion from A_1 to A_2 the lower branch of the loop must be used. From this it follows that we may take the hysteresis effect into account by superposing on the statical stresses, determined from Hooke's law, additional positive stresses on the fibers below the horizontal diameter A_1A_2, and additional negative stresses on the fibers above A_1A_2. This system of stresses corresponds to bending of the shaft in yz plane. Physically these stresses represent bending stresses produced by a force Q which must be applied to the shaft if rotation of the plane of the deflection curve is to be prevented when the shaft is rotating.

From this discussion follows that while the shaft is bent in the xz plane the bending stresses do not produce a bending moment in the same plane but in a plane inclined to the xz plane. In other words, the neutral axis with respect to stresses does not coincide with the neutral axis n–n for strains, but assumes a position n_1n_1 slightly inclined to nn. The same conclusion can be drawn in another way. If we consider a fiber at A_2 moving toward position A_1 the stress will be changing from compression to tension so that the upper portion of the loop in Fig. 130 must be used; from this we see that for zero strain, corresponding to the position of the fiber at A_3, there is a tension stress. In the same way considering the lower branch of the loop, we find that at A_4 there is a compressive stress, thus the vertical diameter A_3A_4, corresponding to points with zero strains, does not any longer represent the neutral axis with respect to stress and the latter must have an inclined position as, for instance, n_1n_1.

In order to get an idea of the magnitude of the force Q we observe that some energy is dissipated during the rotation of the shaft due to hysteresis. Hence a constant torque must act to maintain the constant speed of rotation of the disc. This torque is balanced by the couple represented by the force Q and the corresponding reactions Q' at the bearings, Fig. 130*b*. In this case the work done by the torque during one revolution of the shaft is

$$2\pi Q\delta. \tag{b}$$

This work must be equal to the energy dissipated per cycle due to hysteresis. Unfortunately there is not sufficient information in regard to the area of the hysteresis loop, but it is usually assumed that it does not depend on the frequency. It is also sometimes assumed that it is proportional to the square of the limiting strain,* i.e., in our case, that the dissipation per cycle can be taken in the form

$$E = 2\pi D\delta^2,$$

where D is a constant depending on the hysteresis characteristic of the material of the shaft.

Comparing (*b*) and (*c*) we find

$$Q = D\delta, \tag{d}$$

i.e., the force required to prevent rotation of the deflection curve is proportional to the deflection δ, produced by a static load.

If the shaft is horizontal, it will deflect in a vertical plane due to the gravity force W of the disc, Fig. 131. By applying torque to the disc we can bring the shaft into rotation and we shall find that, owing to hysteresis, the plane of bending takes a slightly inclined position defined by the angle φ. The gravity force W together with the vertical reactions at the bearings form a couple with an arm c balancing the torque applied to the disc. This torque supplies the energy dissipated owing to hysteresis.†

After this preliminary discussion let us derive the differential equations of motion of the center of gravity of the disc on the vertical rotating shaft, assuming: (1) that the speed ω of the rotating shaft is greater than ω_{cr} (2) that the plane of the deflection curve of the shaft is free to rotate with respect to the axis z, Fig. 132; (3) that there is a torque acting on the disc so as to maintain the constant angular velocity ω of the shaft, and (4) that the disc is perfectly balanced and its center of gravity is on the axis of the shaft. Taking, as before, the xy plane as the middle plane of the disc and letting the z axis coincide with the unbent axis of the shaft, we assume, Fig. 132, that the center of the cross-section of the bent shaft coinciding with the center of gravity of the disc is at C, so that $\overline{OC} = \delta$ represents

* See papers by A. L. Kimball and D. E. Lovell, Trans. Am. Soc. Mech. Engrs., v. 48, p. 479, 1926.

† The phenomenon of lateral deflection of a loaded rotating shaft due to hysteresis has been investigated and fully explained by W. Mason—see Engineering, v. 115, p. 698, 1923.

the deflection of the shaft. The angle α between $\overline{OC}$ and the x axis defines the instantaneous position of the rotating plane of the deflection curve of the shaft. We take also some fixed radius $\overline{CB}$ of the shaft and define its angular position during uniform rotation in counter-clockwise direction by the angle ωt measured from the x axis. In writing the differential equations of motion* of the center C we must consider the reaction $k\delta$ of the deflected shaft in the radial direction towards the axis 0, and also the addition reaction Q in tangential direction due to hysteresis. This later reaction is

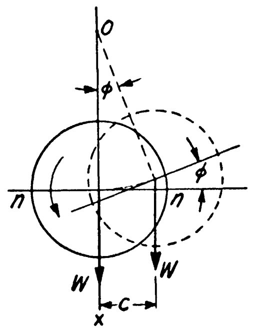

FIG. 131.

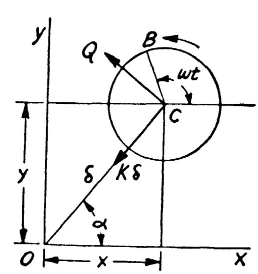

FIG. 132.

evidently equal and opposite to the force Q in Fig. 130b, which was required to prevent the plane of the shaft deflection from rotating. We assume here that $\omega > \dot{\alpha}$, so that the radius $\overline{BC}$ rotates with respect to $\overline{OC}$ in a counter-clockwise direction. Only on this assumption the force Q has the direction shown in Fig. 132 and tends to maintain the rotation of the $\overline{OC}$ plane in a counter-clockwise direction. Denoting by m the mass of the disc, and resolving the forces along the x and the y axes we obtain the following two equations:

$$
\begin{aligned}
m\ddot{x} &= -\, k\delta \cos \alpha - Q \sin \alpha \\
m\ddot{y} &= -\, k\delta \sin \alpha + Q \cos \alpha.
\end{aligned}
\tag{e}
$$

Substituting for Q its expression (d) the equations can be written in the following form:

$$
\begin{aligned}
m\ddot{x} + kx + Dy &= 0 \\
m\ddot{y} + ky - Dx &= 0.
\end{aligned}
\tag{f}
$$

* The discussion of this problem is given in J. G. Baker's paper, loc. cit., p. 110. The consideration of the hysteresis effect in the problem of shaft whirling is introduced first by A. L. Kimball—see Phys. Rev., June, 1923, and Phil. Mag., ser. 6, v. 49, p. 724, 1925.

In solving these equations we assume that:

$$x = Ce^{st}, \qquad y = C'e^{st},$$

and we find in the usual way a biquadratic equation of which the roots are

$$s_{1,2,3,4} = \pm \sqrt{\frac{-k \pm Di}{m}}.$$

Introducing the notation

$$\sqrt{\frac{-k + Di}{m}} = n + p_1 i,$$

from which

$$\left.\begin{aligned} n &= +\sqrt{\frac{-k + \sqrt{k^2 + D^2}}{2m}} \\ p_1 &= +\sqrt{\frac{k + \sqrt{k^2 + D^2}}{2m}} \end{aligned}\right. \tag{g}$$

we can represent the general solution of eqs. (f) in the following form:

$$\begin{aligned} x &= e^{nt}(-C_1 \sin p_1 t + C_2 \cos p_1 t) + e^{-nt}(C_3 \sin p_1 t - C_4 \cos p_1 t) \\ y &= e^{nt}(C_1 \cos p_1 t + C_2 \sin p_1 t) + e^{-nt}(C_3 \cos p_1 t + C_4 \sin p_1 t). \end{aligned} \tag{h}$$

In discussing this solution we must keep in mind that for a material such as steel the tangential force Q is very small in comparison with the radial force $k\delta$. Hence the quantity D is small in comparison with k and we find, from eqs. (g), that n is a small quantity approximately equal to $D/2\sqrt{km}$, while

$$p_1 \approx \sqrt{\frac{k}{m}} = \omega_{cr}.$$

Neglecting the second terms in expressions (h) which will be gradually damped out, and representing the trigonometrical parts of the first terms by projections on the x and the y axes of vectors C_1 and C_2 rotating with the speed ω_{cr}, Fig. 133, we conclude that the shaft is whirling with constant speed ω_{cr} in a counter-clockwise direction while its deflection, equal to $\delta = \sqrt{x^2 + y^2} = e^{nt}\sqrt{C_1{}^2 + C_2{}^2}$, is increasing indefinitely.

It should be noted, however, that in the derivation of eqs. (e) damping forces such as air resistance were entirely neglected. The effect of these forces may increase with the deflection of the shaft so that we may finally obtain a steady whirling of the shaft with the speed approximately equal to ω_{cr}.

In the case of a built-up rotor any friction between the parts of the rotor during bending may have exactly the same effect on the whirling of the rotor as the hysteresis of the shaft material in our previous discussion. If a sleeve or a hub is fixed to a shaft, Fig. 134*a*, and subjected to reversal of bending, the surface fibers of the shaft must slip inside the hub as they elongate and shorten during bending so that some energy of dissipation due to friction is produced. Sometimes the amount of energy dissipated

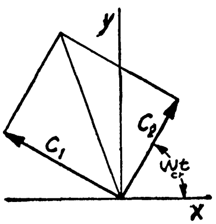

FIG. 133.

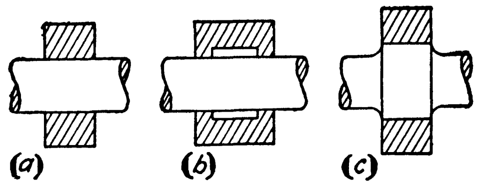

FIG. 134.

owing to such friction is much larger than that due to hysteresis of the material and may cause whirling of rotors running above their critical speeds.* To reduce the effect of friction the dimension of the hub in the axial direction of the shaft must be as short as possible, the construction, in Fig. 134*b*, with bearing surfaces at the ends only, should be avoided. An improvement is obtained by mounting the hub on a boss solid with the shaft, Fig. 134*c*, having large fillets in the corners.

40. Vibrations of Vehicles.—*General Equations.*—The problem of the vibration of a four wheel vehicle as a system with many degrees of freedom is a very complicated one. In the following pages this problem is simplified and only the pitching motion in one plane † (Fig. 135) will be considered. In such a case the system has only *two degrees of freedom* and its position during the vibration can be specified by two coordinates: the vertical displacement z of the center of gravity C and the angle of rotation θ as shown in Fig. 135*b*. Both of these coordinates will be measured from the position of equilibrium.

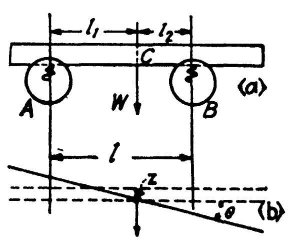

FIG. 135.

* B. L. Newkirk, General Electric Review, vol. 27, p. 169, 1924.

† Rolling motion of the car is excluded from the following discussion.

Let

W be the spring-borne weight of the vehicle.
$I = (W/g)i^2$ be the moment of inertia of the sprung mass about the axis through the center of gravity C.
i be the radius of gyration.
k_1, k_2 are spring constants for the axies A and B, respectively.
l_1, l_2 are distances of the center of gravity from the same axes.*

Then the kinetic energy of motion will be

$$T = \frac{1}{2}\frac{W}{g}\dot{z}^2 + \frac{1}{2}\frac{W}{g}i^2\dot{\theta}^2. \tag{a}$$

In calculating the potential energy, let δ_a, δ_b denote the initial deflections of the springs at the axles A and B, respectively, then,

$$\delta_a = \frac{Wl_2}{lk_1}, \qquad \delta_b = \frac{Wl_1}{lk_2}. \tag{b}$$

The increase in the potential energy of deformation of the springs during motion will be

$$V_1 = \frac{k_1}{2}\{(z - l_1\theta) + \delta_a\}^2 + \frac{k_2}{2}\{(z + l_2\theta) + \delta_b\}^2 - \frac{k_1\delta_a^2}{2} - \frac{k_2\delta_b}{2}$$

or by using (b)

$$V_1 = \frac{k_1}{2}(z - l_1\theta)^2 + \frac{k_2}{2}(z + l_2\theta)^2 + Wz.$$

The decrease in the potential energy of the system due to the lowering of the center of gravity will be

$$V_2 = Wz.$$

The complete expression for the potential energy of the system during motion is therefore

$$V = V_1 - V_2 = \frac{k_1}{2}(z - l_1\theta)^2 + \frac{k_2}{2}(z + l_2\theta)^2. \tag{c}$$

Substituting (a) and (c) in Lagrange's eqs. (73) the following equations for the *free vibrations* of the vehicle will be obtained

* These distances are considered as constant in the further discussion.

$$\frac{W}{g}\ddot{z} = -k_1(z - l_1\theta) - k_2(z + l_2\theta),$$

$$\frac{W}{g}i^2\ddot{\theta} = l_1k_1(z - l_1\theta) - l_2k_2(z + l_2\theta).$$

Letting

$$\frac{(k_1 + k_2)g}{W} = a; \qquad \frac{(-k_1l_1 + k_2l_2)g}{W} = b; \qquad \frac{(l_1^2k_1 + l_2^2k_2)g}{W} = c, \quad (d)$$

we have

$$\ddot{z} + az + b\theta = 0$$

$$\ddot{\theta} + \frac{b}{i^2}z + \frac{c}{i^2}\theta = 0 \qquad (e)$$

These two simultaneous differential equations show that in general the coordinates z and θ are not independent of each other and if, for instance, in order to produce vibrations, the frame of the car be displaced parallel to itself in the z direction and then suddenly released, not only a vertical displacement z but also a rotation θ will take place during the subsequent vibration. The coordinates z and θ become independent only in the case when $b = 0$ in eqs. (e). This occurs when

$$k_1l_1 = k_2l_2, \qquad (f)$$

i.e., when the spring constants are inversely proportional to the spring distances from the center of gravity. In such cases a load applied at the center of gravity will only produce vertical displacement of the frame without rotation. Such conditions exist in the case of railway carriages where usually $l_1 = l_2$ and $k_1 = k_2$.

Returning now to the general case we take the solution of the eqs. (e) in the following form

$$z = A\cos(pt + \alpha); \qquad \theta = B\cos(pt + \alpha).$$

Substituting in eqs. (e) we obtain

$$A(a - p^2) + bB = 0, \qquad (g)$$

$$\frac{b}{i^2}A + \left(\frac{c}{i^2} - p^2\right)B = 0.$$

Eliminating A and B from eqs. (g) the following frequency equation will be obtained,

$$(a - p^2)\left(\frac{c}{i^2} - p^2\right) - \frac{b^2}{i^2} = 0. \tag{h}$$

The two roots of eq. (h) considered as an equation in p^2 are

$$p^2 = \frac{1}{2}\left(\frac{c}{i^2} + a\right) \pm \sqrt{\frac{1}{4}\left(\frac{c}{i^2} + a\right)^2 - \frac{ac}{i^2} + \frac{b^2}{i^2}}$$

$$= \frac{1}{2}\left(\frac{c}{i^2} + a\right) \pm \sqrt{\frac{1}{4}\left(\frac{c}{i^2} - a\right)^2 + \frac{b^2}{i^2}}. \tag{k}$$

Noting that from eq. (d),

$$ac - b^2 = \frac{g^2}{W^2} k_1 k_2 (l_1 + l_2)^2,$$

it can be concluded that both roots of eq. (h) are real and positive.

Principal Modes of Vibration.—Substituting (k) in the first of the eqs. (g) the following values for the ratio A/B between the amplitudes will be obtained.

$$\frac{A}{B} = \frac{b}{p^2 - a} = \frac{b}{\frac{1}{2}\left(\frac{c}{i^2} - a\right) \pm \sqrt{\frac{1}{4}\left(\frac{c}{i^2} - a\right)^2 + \frac{b^2}{i^2}}}. \tag{l}$$

The $+$ sign, as is seen from (k), corresponds to the mode of vibration having the higher frequency while the $-$ sign corresponds to vibrations of lower frequency.

In the further discussion it will be assumed that

$$b > 0 \quad \text{or} \quad k_2 l_2 > k_1 l_1.$$

This means that under the action of its own weight the displacement of the car is such as shown in Fig. 136; the displacement in downward direction is associated with a rotation in the direction of the negative θ. Under this assumption the amplitudes A and B will have opposite signs if the negative sign be taken before the radical in the denominator of (l) and they will have the same signs when the positive sign be taken. The corresponding two types of vibration are shown in Fig. 137. The type (a)

has a lower frequency and can be considered as a rotation about a certain point Q to the right of the center of gravity C. The type (b) having a higher frequency, consists of a rotation about a certain point P to the left of C. The distances m and n of the points Q and P from the center of

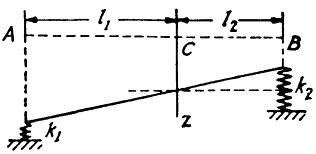

FIG. 136.

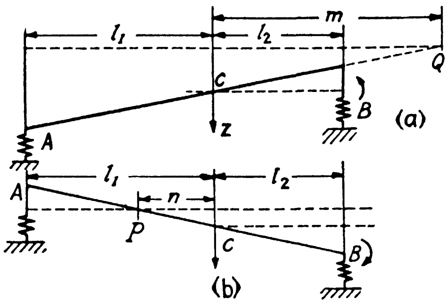

FIG. 137.

gravity are given by the absolute values of the right side of eq. (l) and we obtain a very simple relation,

$$mn = -\frac{b}{\frac{1}{2}\left(\frac{c}{i^2} - a\right) + \sqrt{\frac{1}{4}\left(\frac{c}{i^2} - a\right)^2 + \frac{b^2}{i^2}}} \cdot \frac{b}{\frac{1}{2}\left(\frac{c}{i^2} - a\right) - \sqrt{\frac{1}{4}\left(\frac{c}{i^2} - a\right)^2 + \frac{b^2}{i^2}}} = i^2. \qquad (m)$$

In the particular case, when $b = 0$, i.e., $k_1l_1 = k_2l_2$ the distance n becomes equal to zero and m becomes infinitely large. This means that in this case one of the principal modes of vibration consists of a rotation about the center of gravity and the other consists of a translatory movement without rotation. A vertical load applied at the center of gravity in this case will produce only a vertical displacement and both springs will get equal compressions.

If, in addition to b, $(c/i^2) - a$ becomes equal to zero, both frequencies, as given by eq. (k), become equal and the two types of vibration will have the same period.

Numerical Example.—A numerical example of the above theory will now be considered.* Taking a case with the following data: $W = 966$ lbs.;

* See the paper by H. S. Rowell, Proc. Inst. Automobile Engineers, London, Vol. XVII, Part II, p. 455 (1923).

$i^2 = 13$ ft.^{2}; $l_1 = 4$ ft.; $l_2 = 5$ ft.; $k_1 = 1600$ lbs./ft.; $k_2 = 2400$ lb./ft., the corresponding static deflections (see eq. (b)) are

$$\delta_a = 4.0 \text{ in.}, \qquad \delta_b = 2.15 \text{ in.}$$

From eqs. (d)

$$a = 133.3, \qquad b = 186.7, \qquad c = 2853.$$

Substituting in (k) we obtain the following two roots $p_1{}^2 = 109$, $p_2{}^2 = 244$. The corresponding frequencies are

$p_1 = 10.5$ radians per second and $p_2 = 15.6$ radians per second, respectively, or

$N_1 = 100$ and $N_2 = 150$ complete oscillations per minute.

From eq. (l) we have

$$\frac{A}{B} = -\ 7.71 \text{ ft.} \qquad \text{and} \qquad \frac{A}{B} = 1.69 \text{ ft.}$$

This means that in the slower mode of vibration the sprung weight oscillates 7.71 ft. per radian of pitching motion or 1.62 inches per degree.

In the higher mode of vibration the sprung weight oscillates 1.69 ft. for every radian of pitching motion or .355 inch per degree.

Roughly speaking in the slower mode of vibration the car is bouncing, the deflections of two springs being of the same sign and in the ratio

$$\frac{\delta_b'}{\delta_a'} = \frac{7.71 - 5}{7.71 + 4} = .23.$$

In the quicker mode of vibration the car is mostly pitching.

It is interesting to note that a good approximation for the frequencies of the principal modes of vibration can be obtained by using the theory of a system with one degree of freedom. Assuming first that the spring at B (see Fig. 135) is removed so that the car can bounce on the spring A about the axis B as a hinge. Then the equation of motion is

$$\left(I + \frac{W}{g} l_2{}^2\right) \ddot{\theta} + k_1 l^2 \theta = 0,$$

so that the "constrained" frequency is

$$p_1' = l \sqrt{\frac{k_1}{I + \dfrac{W}{g} l_2{}^2}},$$

or substituting the numerical data of the above example

$$p_1' = 9\sqrt{\frac{1600}{\frac{966}{32.2}(13 + 25)}} = 10.7.$$

This is in good agreement with the frequency 10.5 obtained above for the lower type of vibration of the car. In the same manner considering the bouncing of the car on the spring B about the axis A as a hinge, we obtain $p_2' = 15.0$ as compared with $p_2 = 15.6$ given above for the quicker mode of vibration.

On the basis of this a practical method for obtaining the frequencies of the principal modes of vibration by test is to lock the front springs and bounce the car; then lock the rear springs and again bounce the car. The frequencies obtained by these tests will represent a good approximation.

Beating Phenomena.—Returning now to the general solution of the eqs. (e) and denoting by p_1 and p_2, the two roots obtained from (k) we have

$$\begin{aligned} z &= A_1 \cos (p_1t + \alpha_1) + A_2 \cos (p_2t + \alpha_2), \\ \theta &= B_1 \cos (p_1t + \alpha_1) + B_2 \cos (p_2t + \alpha_2), \end{aligned} \tag{r}$$

in which (see eq. (l))

$$\frac{A_1}{B_1} = \frac{b}{p_1^2 - a}; \qquad \frac{A_2}{B_2} = \frac{b}{p_2^2 - a}. \tag{s}$$

The general solution (r) contains four arbitrary constants A_1, A_2, α_1, and α_2, which must be determined for every particular case so as to satisfy the initial conditions. Assume, for instance, that in the initial moment a displacement λ exists in a downward direction without rotation and that the car is then suddenly released. In such a case the initial conditions are

$$(z)_{t=0} = \lambda; \qquad (\dot{z})_{t=0} = 0; \qquad (\theta)_{t=0} = 0; \qquad (\dot{\theta})_{t=0} = 0.$$

These conditions will be satisfied by taking in eqs. (r)

$$\alpha_1 = \alpha_2 = 0,$$

$$A_1 = \lambda\frac{a - p_2^2}{p_1^2 - p_2^2}; \qquad A_2 = \lambda\frac{p_1^2 - a}{p_1^2 - p_2^2};$$

$$B_1 = A_1\frac{(p_1^2 - a)}{b}; \qquad B_2 = A_2\frac{(p_2^2 - a)}{b}. \tag{t}$$

We see that under the assumed conditions both modes of vibration will be produced which at the beginning will be in the same phase but with elapse of time, due to the difference in frequencies, they will become displaced with respect to each other and a complicated combined motion will take place. If the difference of frequencies is a very small one the characteristic "*beating phenomenon*," i.e., vibrations with periodically varying amplitude, will take place. In considering this particular case, assume in eq. (k) that

$$\frac{c}{i^2} - a = 0 \qquad \text{and} \qquad \frac{b}{i} = \delta,$$

where δ is a small quantity. Then

$$p_1^2 = a - \delta; \qquad p_2^2 = a + \delta,$$

and from (t) we obtain,

$$A_1 = \frac{\lambda}{2}; \qquad A_2 = \frac{\lambda}{2}; \qquad B_1 = -\frac{\lambda}{2i}; \qquad B_2 = \frac{\lambda}{2i}.$$

Solution (n) becomes

$$z = \frac{\lambda}{2}(\cos p_1 t + \cos p_2 t) = \lambda \cos \frac{p_1 + p_2}{2} t \cos \frac{p_1 - p_2}{2} t,$$

$$\theta = \frac{\lambda}{2i}(-\cos p_1 t + \cos p_2 t) = \frac{\lambda}{i} \sin \frac{p_1 + p_2}{2} t \sin \frac{p_1 - p_2}{2} t. \qquad (u)$$

Owing to the fact that $p_1 - p_2$ is a small quantity the functions $\cos\{(p_1 + p_2)/2\}t$ and $\sin\{(p_1 + p_2)/2\}t$ will be quickly varying functions so that they will perform several cycles before the slowly varying function $\sin\{(p_1 - p_2)/2\}t$ or $\cos\{(p_1 - p_2)/2\}t$ can undergo considerable change. As a result, oscillations with periodically varying amplitudes will be obtained (see Fig. 12).

Forced Vibrations.—The disturbing forces producing forced oscillations of a car are transmitted by the springs. In the general discussion above it was shown that the two principal modes of vibration are oscillations about two definite points P and Q (Fig. 137). The corresponding generalized forces in such a case are the moments of the spring forces about the points P and Q. From this it can be concluded that any fluctuation in a spring force, produced by some kind of unevenness of the road, will produce simultaneously both types of vibrations provided that this spring force does not pass through one of the points P or Q. Assume, for instance, that the front wheels of a moving car encounter an obstacle on the road, the corresponding compression of the front springs will produce vibrations of the car. Now when the rear wheels reach the same obstacle, an additional impulse will be given to the oscillating car. The oscillations produced by this new impulse will be superimposed on the previous oscillations and the resulting motion will depend on the value Δt of the interval of time between the two impulses, or, denoting by v the velocity of the car, on the magnitude of l/v. It is easy to see that at a certain value of v the effects of the two impulses will be added and we will get very unfavorable conditions for these *critical speeds*. Let τ_1 and τ_2 denote the periods of the two principal modes of vibration and assume the interval $\Delta t = (l/v)$ be a multiple of these periods, so that

$$\Delta t = m_1\tau_1 = m_2\tau_2,$$

where m_1 and m_2 are integer numbers. Then the impulses will repeat after an integer number of oscillations and *resonance conditions* will take place.* Under such conditions large oscillations may be produced if there is not enough friction in the springs.

From this discussion it is clear that an arrangement where an impulse produced by one spring does not affect the other spring may be of practical interest. This condition will be satisfied when the body of the car can be replaced by a dynamical model with two masses W_1 and W_2 (Fig. 138) concentrated at the springs A and B. In this case we have

$$W_1 = \frac{Wl_2}{l}; \qquad W_2 = \frac{Wl_1}{l},$$

$$W_1l_1^2 + W_2l_2^2 = Wi^2,$$

from which

$$l_1l_2 = i^2. \tag{80}$$

Comparing with eq. (m) it can be concluded that the points P and Q (see Fig. 137) coincide in this case with the points A and B so that the fluctuations in the spring forces will be independent of each other and the condition of resonance will be excluded. It should be noted that when $l_1 = l_2$ condition (80) coincides with the rule given by Prof. H. Reissner that the radius of gyration of the sprung mass should be half the wheel

* See P. Lemaire, La Technique Moderne, January 1921. See also the paper by H. S. Rowell, p. 481, mentioned above.

base. In most of the modern cars the wheel base is larger than that given by eq. (80). This discrepancy should be attributed to steering and skidding conditions which necessitate an increase in wheel base.

Pressure on the Road.—Due to dynamical causes the pressure of a wheel on the road during motion will be usually different from what we would have in the statical condition. Assuming the simple case illustrated in Fig. 138, the pressure of the wheel can be found from a consideration of the motion of the system, shown in Fig. 139, in which W_1 is weight

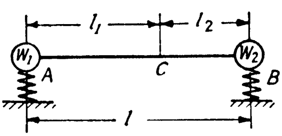

FIG. 138.

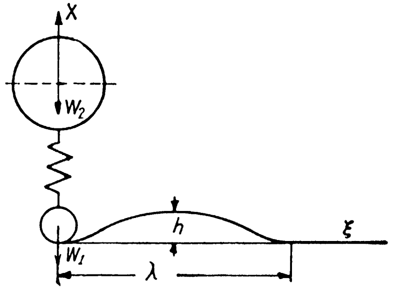

FIG. 139.

directly transmitted on the road,* W_2 is spring borne weight, v is constant velocity of the motion of the wheel along the horizontal axis, x_1, x_2 are displacements in an upward direction of the weights W_1 and W_2 from their position of equilibrium shown in Fig. 139. If there is no unevenness of the road, no vibration will take place during motion and the pressure on the road will be equal to the statical. Assume now that the road contour is rigid and can be represented by the equation:

$$x = \frac{h}{2}\left(1 - \cos\frac{2\pi\xi}{\lambda}\right),$$

where ξ is measured along the horizontal axis and λ is the wave length. During rolling with a constant velocity v along these waves the vertical displacements of the wheel considered as rigid will be represented by the equation

$$x_1 = \frac{h}{2}\left(1 - \cos\frac{2\pi vt}{\lambda}\right). \qquad (a)$$

The corresponding acceleration in a vertical direction is

$$\ddot{x} = \frac{h}{2}\frac{4\pi^2v^2}{\lambda^2}\cos\frac{2\pi vt}{\lambda}.$$

* Spring effect of the tire is neglected in this discussion.

Adding the inertia force to the weight the pressure on the road due to the unsprung mass alone will be

$$W_1 + \frac{W_1}{g}\frac{h}{2}\frac{4\pi^2 v^2}{\lambda^2}\cos\frac{2\pi vt}{\lambda}. \tag{b}$$

The maximum pressure occurs when the wheel occupies the lowest position on the contour and is equal to

$$W_1 + \frac{W_1}{g}\frac{h}{2}\frac{4\pi^2 v^2}{\lambda^2}.$$

It is seen that the *dynamical effect* due to the inertia force increases as the square of the speed.

In order to obtain the complete pressure on the road, the pressure due to the spring force must be added to pressure (*b*) calculated above. This force will be given by the expression

$$W_2 - k(x_2 - x_1), \tag{c}$$

in which the second term represents the change in the force of the spring due to the relative displacement $x_2 - x_1$ of the masses W_1 and W_2. This displacement can be obtained from the differential equation

$$\frac{W_2}{g}\ddot{x}_2 + k(x_2 - x_1) = 0, \tag{d}$$

representing the equation of motion of the sprung weight W_2.

Substituting (*a*) for x_1 we have

$$\frac{W_2}{g}\ddot{x}_2 + kx_2 = \frac{kh}{2}\left(1 - \cos\frac{2\pi vt}{\lambda}\right). \tag{e}$$

This equation represents vibration of the sprung weight produced by the wavy contour of the road. Assuming that at the beginning of the motion $x_1 = x_2 = 0$ and $\dot{x}_1 = \dot{x}_2 = 0$, the solution of eq. (*e*) will be

$$x_2 = \frac{h}{2}\left(1 + \frac{\tau_1^2}{\tau_2^2 - \tau_1^2}\cos\frac{2\pi t}{\tau_1} - \frac{\tau_2^2}{\tau_2^2 - \tau_1^2}\cos\frac{2\pi t}{\tau_2}\right), \tag{f}$$

in which

$\tau_1 = 2\pi\sqrt{(W_2/kg)}$ natural period of vibration of the sprung weight,

$\tau_2 = (\lambda/v)$ time necessary to cross the wave length λ.

The force in the spring, from eqs. (a) and (c), is

$$W_2 - \frac{kh}{2}\frac{\tau_1^2}{\tau_2^2 - \tau_1^2}\left(\cos\frac{2\pi t}{\tau_1} - \cos\frac{2\pi t}{\tau_2}\right). \qquad (g)$$

Now, from (b) and (g), the pressure on the road in addition to the statical pressure will be

$$\frac{W_1}{g}\frac{h}{2}\frac{4\pi^2}{\tau_2^2}\cos\frac{2\pi t}{\tau_2} - \frac{kh}{2}\frac{\tau_1^2}{\tau_2^2 - \tau_1^2}\left(\cos\frac{2\pi t}{\tau_1} - \cos\frac{2\pi t}{\tau_2}\right). \qquad (h)$$

The importance of the first term increases with the speed while the second term becomes important under conditions of resonance. On this basis it can be concluded that with a good road surface and high speed the unsprung mass decides the road pressure and in the case of a rough road the sprung mass becomes important.

41. Dynamic Vibration Absorber.—In discussing forced vibration of systems with one degree of freedom it was shown how the amplitude of this vibration can be reduced by a proper choice of the spring constant so that the system will be far away from resonance, or by a proper balancing which minimizes the magnitude of the disturbing force. Sometimes these methods are impractical and a special device for reducing vibrations, called the *dynamic vibration absorber*, must be used. An example of such a device is illustrated in Fig. 140. A machine or a machine part under consideration is represented by a weight W_1, Fig. 140a, suspended on a spring having the spring constant k_1. The natural frequency of vibration of this system is

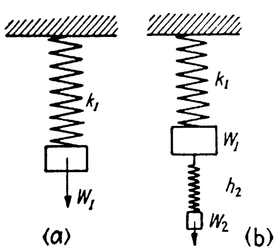

FIG. 140.

$$p = \sqrt{\frac{k_1 g}{W_1}}. \qquad (a)$$

If a pulsating force $P \cos \omega t$ is acting vertically on the weight W_1, forced vibration will be produced of a magnitude

$$x_1 = \frac{P}{k_1} \cdot \frac{1}{1 - \omega^2/p^2}\cos \omega t. \qquad (b)$$

This vibration may become very large when the ratio p/ω approaches unity. To reduce the vibration, let us attach a small weight W_2 to the machine

W_1 by a spring having a spring constant k_2, Fig. 140*b*. It will be shown in our further discussion that by a proper choice of the weight W_2 and of the spring constant k_2 a substantial reduction in vibration of the main system, Fig. 140*a*, can be accomplished. The attached system consisting of weight W_2 and spring k_2 is a dynamical vibration absorber.

*The Absorber without Damping.**—To simplify the discussion let us assume first that there is no damping in the system. By attaching the

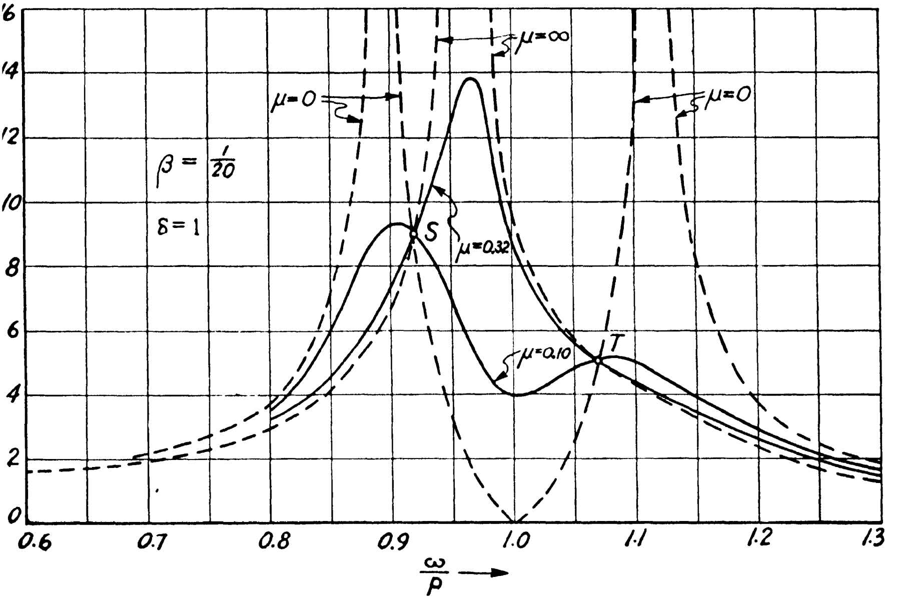

FIG. 141.

vibration absorber to the main system we obtain a system with two degrees of freedom. As coordinates of the system we take vertical displacements x_1 and x_2 of the weights W_1 and W_2 from their positions of static equilibrium. The downward directions of these displacements are taken positive. If the mass of the springs be neglected, the kinetic energy of the system is

$$T = \frac{1}{2g}(W_1\dot{x}_1^2 + W_2\dot{x}_2^2). \tag{c}$$

* See paper by J. Ormondroyd and J. P. Den Hartog, Trans. Amer. Soc. Mech. Engrs., v. 50, no. 7, p. 9, 1928. See also H. Holzer, Stodola's Festschrift, p. 234, 1929, Zürich.

Observing that x_1 and x_2-x_1 are the elongations of the upper and of the lower springs respectively, the potential energy of the system, calculated from the position of equilibrium, is

$$V = \tfrac{1}{2}[k_1x_1^2 + k_2(x_2 - x_1)^2] \tag{d}$$

Substituting in Lagrange's equation, we obtain

$$\begin{aligned} &\frac{W_1}{g}\ddot{x}_1 + k_1x_1 - k_2(x_2 - x_1) = P\cos\omega t \\ &\frac{W_2}{g}\ddot{x}_2 + k_2(x_2 - x_1) = 0. \end{aligned} \tag{e}$$

The same equations can be readily obtained by writing the equation of motion for each mass considering it a particle and observing that on the lower mass the force $-k_2(x_2 - x_1)$ and on the upper mass the forces $-k_1x_1$, $k_2(x_2 - x_1)$ and $P\cos\omega t$ are acting.

The steady state of the forced vibration will be obtained by taking solutions of equations (*e*) in the following form:

$$x_1 = \lambda_1\cos\omega t, \qquad x_2 = \lambda_2\cos\omega t. \tag{f}$$

Substituting these expressions in (*e*) we obtain the following expressions for the quantities λ_1 and λ_2, the absolute values of which are the amplitudes of the forced vibrations of masses W_1 and W_2.

$$\begin{aligned} \lambda_1 &= \frac{P(k_2 - \omega^2W_2/g)}{(k_1 + k_2 - \omega^2W_1/g)(k_2 - \omega^2W_2/g) - k_2^2} \\ \lambda_2 &= \frac{Pk_2}{(k_1 + k_2 - \omega^2W_1/g)(k_2 - \omega^2W_2/g) - k_2^2}. \end{aligned} \tag{g}$$

To simplify our further discussion we bring these expressions into dimensionless form. For this purpose we introduce the following notations:

$\lambda_{st} = P/k_1$ is the static deflection of the main system produced by the force P.

$p_1 = \sqrt{k_2g/W_2}$ is the natural frequency of the absorber.

$\beta = W_2/W_1$ is the ratio of the weights of the absorber and of the main system.

$\delta = p_1/p$ is the ratio of the natural frequencies of the absorber and of the main system.

$\gamma = \omega/p$ is the ratio of the frequency of the disturbing force to the natural frequency of the main system.

Then, from expressions (g), we obtain

$$\lambda_1/\lambda_{st} = \frac{1 - \omega^2/p_1^2}{(1 + k_2/k_1 - \omega^2/p^2)(1 - \omega^2/p_1^2) - k_2/k_1}.$$

$$\lambda_2/\lambda_{st} = \frac{1}{(1 + k_2/k_1 - \omega^2/p^2)(1 - \omega^2/p_1^2) - k_2/k_1}. \tag{h}$$

The first of these expressions we represent also in the following form

$$\lambda_1/\lambda_{st} = \frac{\gamma^2 - \delta^2}{\beta\delta^2\gamma^2 - (\gamma^2 - 1)(\gamma^2 - \delta^2)}. \tag{h}'$$

It is seen from the first of expressions (h) that the motion of the main mass vanishes if we take

$$\omega^2 = p_1^2 = \frac{k_2 g}{W_2}, \tag{i}$$

i.e., if we select the proportions of the absorber so as to make its natural frequency equal to the frequency of the pulsating force. Then from the second eq. (h) we find

$$\lambda_2 = -\frac{\lambda_{st}\, k_1}{k_2} = -\frac{P}{k_2}$$

and the vibrations of the two masses, from (f), are

$$x_1 = 0, \qquad x_2 = -\frac{P}{k_2}\cos\omega t. \tag{j}$$

We see that the weight W_2 of the absorber moves in such a way that the spring force $k_2x_2 = -P\cos\omega t$, acting on the machine W_1, is always equal and opposite to the impressed force, thus the motion of W_1 is eliminated completely.

In designing an absorber we must satisfy the condition (i) from which the ratio k_2/W_2 is obtained provided that the constant frequency of the pulsating force is known. The absolute values of the quantities k_2 and W_2 are also of practical importance. We see from the second of eqs. (j) that if k_2 is taken too small, x_2 becomes large and the stress in the spring may become excessive. Thus equations (i) and (j) must both be considered in the practical design of an absorber and the smallest possible values of k_2 and W_2 will depend on the maximum value of the pulsating force P and on the allowable travel of the weight W_2.

So far the action of the absorber has been discussed for one frequency of the pulsating force only, namely for that satisfying eq. (i). For any other frequency both masses, W_1 and W_2, will vibrate and the amplitudes of these vibrations are obtained from eqs. (h). We therefore have a system with two degrees of freedom and with two critical values of ω, corresponding to the two conditions of resonance. These critical values are obtained by equating the denominator of expressions (h) to zero. In this way we find

$$(1 + k_2/k_1 - \omega^2/p^2)(1 - \omega^2/p_1^2) - k_2/k_1 = 0. \qquad (k)$$

From this quadratic equation in ω^2, the two critical frequencies can be calculated in each particular case. The amplitudes of vibration of the weight W_1 will be calculated from eq. (h) for any value of the ratio ω/p and can be represented graphically. For a particular case when $p = p_1$ and $W_2/W_1 = k_2/k_1 = .05$, the amplitudes are shown in Fig. 141 by the dotted line curves (resonance curves) marked $\mu = 0$. In this particular case zero amplitude of the main mass W_1 is obtained when $\omega = p_1 = p$. The amplitudes increase indefinitely when the ratio ω/p approaches its critical values $\omega_1/p = .895$ and $\omega_2/p = 1.12$.

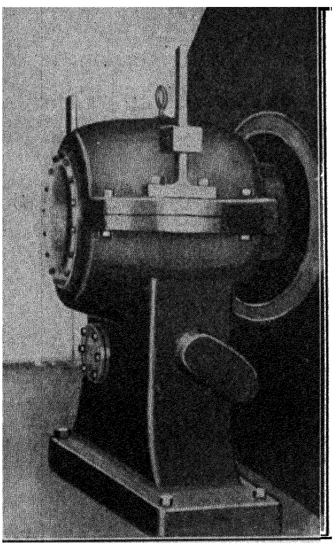

FIG. 142.

From this it is seen that the applicability of the absorber without damping is restricted to machines with constant speed such as for instance electric synchronous or induction machines. One application of the absorber is shown in Fig. 142, which represents the outboard generator bearing pedestal of a 30,000 KW. turbo-generator. This pedestal vibrated considerably at 1800 R.P.M. in the direction of the generator axis. By bolting to the pedestal two vibration absorbers consisting of two cantilevers 20 in. long and 7/8 in. × 2 5/8 in. in cross section, weighted at the end with 25 lbs., the amplitude was reduced to about one third of its previous magnitude.

The described method of eliminating vibration may be used also in the case of torsional systems shown in Fig. 143. A system consisting of two masses with the moments of inertia I_1, I_2 and a shaft with a spring constant k, has a period of natural vibration equal to, see eq. (16),

$$\tau = 2\pi \sqrt{\frac{1}{k} \frac{I_1 I_2}{I_1 + I_2}}. \qquad (16)$$

If a pulsating torque $M_t \cos \omega t$ is acting on the mass I_1 the forced torsional vibration of both discs I_1 and I_2 produced by this torque can be eliminated by attaching to I_1 a small vibrating system consisting of a disc with a moment of inertia i and a shaft with a spring constant k_1 (Fig. 143, *b*). It is only necessary to take for k_1 and i such proportions as to make the frequency of the attached system equal to the frequency of the pulsating torque.

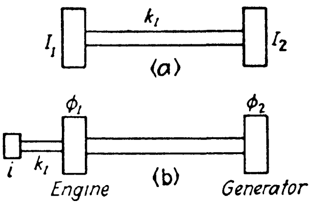

FIG. 143.

*Damped Vibration Absorber.**—In order to make the absorber effective over an extended range of frequencies it is necessary to introduce damping in the vibrating system. Assume that a damping device is located between the masses W_1 and W_2, Fig. 140*b*, and that the magnitude of the damping is proportional to the relative velocity $\dot{x}_1 - \dot{x}_2$. Introducing the friction force into eqs. (*e*) by adding it to the right side, we obtain the equations:

$$\frac{W_1}{g}\ddot{x}_1 + k_1 x_1 - k_2(x_2 - x_1) = P\cos\omega t + c(\dot{x}_2 - \dot{x}_1)$$

$$\frac{W_2}{g}\ddot{x}_2 + k_2(x_2 - x_1) = c(\dot{x}_1 - \dot{x}_2), \tag{l}$$

in which factor c denotes the magnitude of the damping force when the relative velocity between the two masses is equal to unity.

Observing that due to damping there must be a phase difference between the pulsating force and the vibration, we represent the steady forced vibration of the system in the form

$$\begin{aligned} x_1 &= C_1 \cos\omega t + C_2 \sin\omega t \\ x_2 &= C_3 \cos\omega t + C_4 \sin\omega t. \end{aligned} \tag{m}$$

Substituting these expressions into equations (*l*), we obtain four algebraic linear equations for determining the constants $C_1 \cdot \cdot \cdot C_4$. In our further

* See paper by J. Ormondroyd and J. P. Den Hartog, loc. cit., p. 241, also papers by E. Hahnkamm, Annalen d. Physik, 5 Folge, v. 14, p. 683, 1932; Zeitschr. f. angew. Math. and Mech., Vol. 13, p. 183, 1933; Ingenieur-Archiv, v. 4, p. 192, 1933. The effect of internal friction on damping was discussed by O. Foppl, Ing. Archiv, v. 1, p. 223, 1930. See also his book, "Aufschaukelung und Dampfung von Schwingungen," Berlin, 1936, and the paper by G. Bock, Zeitschr. f. angew. Math. u. Mech., V. 12, p. 261, 1932.

discussion we will be interested in the amplitude of forced vibration of the mass W_1 which is equal to

$$(x_1)_{\max} = \lambda_1 = \sqrt{C_1^2 + C_2^2}.$$

Omitting all intervening calculations of the constants C_1 and C_2, and using our previous notations (see p. 242) we obtain:

$$\lambda_1^2/\lambda_{st}^2 = \frac{4\mu^2\gamma^2 + (\gamma^2 - \delta^2)^2}{4\mu^2\gamma^2(\gamma^2 - 1 + \beta\gamma^2)^2 + [\beta\delta^2\gamma^2 - (\gamma^2 - 1)(\gamma^2 - \delta^2)]^2} \tag{n}$$

in which the damping is defined by $\mu = cg/2W_2p$.

From this expression the amplitude of the forced vibration of the weight W_1 can be calculated for any value of $\gamma = \omega/p$ if the quantities δ and β, defining the frequency and the weight of the absorber, and the quantity μ are known.

By taking $\mu = 0$ we obtain from (n) expression $(h)'$ already found before for an absorber without damping. The resonance curves ($\mu = 0$) giving the amplitude of vibration for $\beta = 1/20$, $\delta = 1$, and for various values of $\gamma = \omega/p$ are shown in Fig. 141 by dotted lines. It should be noted that the absolute values of expression $(h)'$ are plotted in the figure, while $(h)'$ is changing sign at $\gamma = .895$, $\gamma = 1$ and $\gamma = 1.12$.

Another extreme case is defined by taking $\mu = \infty$. If damping is infinitely large there will be no relative motion between W_1 and W_2. We obtain then a system with one degree of freedom of the weight $W_1 + W_2$ and with the spring constant k_1. For determining the amplitude of the forced vibration for this system we have, from (n)

$$\lambda_1^2/\lambda_{st}^2 = \frac{1}{(\gamma^2 - 1 + \beta\gamma^2)^2}. \tag{o}$$

The critical frequency for this system is obtained by equating the denominator of expression (o) to zero. Thus

$$\gamma^2 - 1 + \beta\gamma^2 = 0 \tag{p}$$

and

$$\gamma_{cr} = \sqrt{\frac{1}{1 + \beta}}.$$

The resonance curves for $\mu = \infty$ are also shown in Fig. 141 by dotted lines. These curves are similar to those in Fig. 10 (p. 15) obtained before for systems with one degree of freedom. For any other value of (μ) the resonance curves can be plotted by using expression (n). In Fig. 141 the curves for $\mu = 0.10$ and for $\mu = 0.32$ are shown. It is interesting to note that all these curves are intersecting at points S and T. This means that for the two corresponding values of γ the amplitudes of the forced vibration of the weight W_1 are independent of the amount of damping. These values of γ can be found by equating the absolute values of λ_1/λ_{st} as obtained from (o) and from $(h)'$. Thus we have *

$$\frac{\gamma^2 - \delta^2}{\beta\delta^2\gamma^2 - (\gamma^2 - 1)(\gamma^2 - \delta^2)} = \frac{1}{\gamma^2 - 1 + \beta\gamma^2}. \tag{q}$$

The same equation can be deduced from expression (n). The points of intersection S and T define those values of γ for which the magnitude of the expression (n) does not depend on damping, i.e., are independent of μ. The expression (n) has the form

$$\frac{M\mu^2 + N}{P\mu^2 + Q}$$

so that it will be independent of μ^2 only if we have $M/P = N/Q$; this brings us again to eq. (q). This equation can be put into the form

$$(\gamma^2 - \delta^2)(\gamma^2 - 1 + \beta\gamma^2) = \beta\delta^2\gamma^2 - (\gamma^2 - 1)(\gamma^2 - \delta^2)$$

or

$$\gamma^4 - 2\gamma^2 \frac{1 + \delta^2 + \beta\delta^2}{2 + \beta} + \frac{2\delta^2}{2 + \beta} = 0. \tag{r}$$

From this equation two roots γ_1^2 and γ_2^2 can be found which determine the abscissas of the points S and T. The corresponding values of the amplitudes of the forced vibration are obtained by substituting γ_1^2 and γ_2^2 in eq. (n) or in eq. (o). Using the latter as a simpler one, we obtain for the ordinates of points S and T the expressions †

$$-\frac{\lambda_{st}}{\gamma_1^2 - 1 + \beta\gamma_1^2} \quad \text{and} \quad \frac{\lambda_{st}}{\gamma_2^2 - 1 + \beta\gamma_2^2} \tag{s}$$

respectively. The magnitudes of these ordinates depend on the quantities β and δ defining the weight and the spring of the absorber. By a proper choice of these characteristics we can improve the efficiency of the absorber. Since all such curves as are shown in Fig. 141 must pass through the points

* For the point of intersection S both sides of this equation are negative and for the point T-positive as can be seen from the roots of equations (k) and (p).

† It is assumed that γ_1^2 is the smaller root of eq. (r) and the minus sign must be taken before the square root from (o) to get a positive value for the amplitude.

S and T, the maximum ordinates of these curves giving the maximum amplitudes of the forced vibration will depend on the ordinates of points S and T, and it is reasonable to expect that the most favorable condition will be obtained by making the ordinates of S and T equal.* This requires that:

$$-\frac{\lambda_{st}}{\gamma_1^2 - 1 + \beta\gamma_1^2} = \frac{\lambda_{st}}{\gamma_2^2 - 1 + \beta\gamma_2^2}$$

or

$$\gamma_1^2 + \gamma_2^2 = \frac{2}{1 + \beta}. \tag{t}$$

Remembering that γ_1^2 and γ_2^2 are the two roots of the quadratic equation (r) and that for such an equation the sum of the two roots is equal to the coefficient of the middle term with a negative sign, we obtain:

$$\frac{2}{1 + \beta} = 2\frac{1 + \delta^2 + \beta\delta^2}{2 + \beta}$$

from which

$$\delta = \frac{1}{1 + \beta}. \tag{81}$$

This simple formula gives the proper way of " tuning " the absorber. If the weight W_2 of the absorber is chosen, the value of β is known and we determine, from eq. (81), the proper value of δ, which defines the frequency and the spring constant of the absorber.

To determine the amplitude of forced vibrations corresponding to points S and T we substitute in (s) the value of one of the roots of eq. (r). For a properly tuned absorber, eq. (81) holds, and this later equation becomes

$$\gamma^4 - \frac{2\gamma^2}{1 + \beta} + \frac{2}{(2 + \beta)(1 + \beta)^2} = 0 \tag{u}$$

from which

$$\gamma^2 = \frac{1}{1 + \beta}\left(1 \pm \sqrt{\frac{\beta}{2 + \beta}}\right).$$

Then, from (s)

$$\lambda_1/\lambda_{st} = \sqrt{\frac{2 + \beta}{\beta}}. \tag{82}$$

* This question is discussed with much detail in the above-mentioned paper by Hahnkamm, loc. cit., p. 245.

So far the quantity μ defining the amount of damping in the absorber did not enter into our discussion since the position of the points S and T is independent of μ. But the maximum ordinates of the resonance curves passing through the points S and T depend, as we see from Fig. 141, on the magnitude of μ. We shall get the most favorable condition by selecting μ in such a way as to make the resonance curves have a horizontal tangent at S or at T. Two curves of this kind, one having a maximum at S and

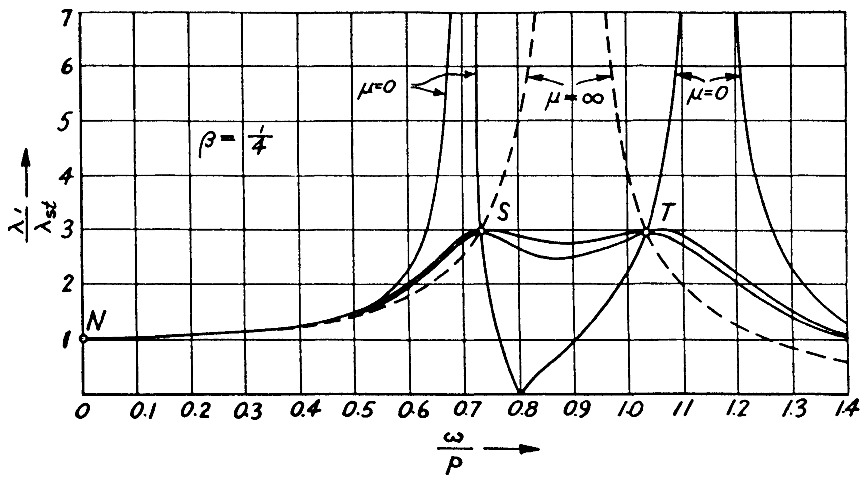

FIG. 144.

the other having a maximum at T are shown in Fig. 144. They are calculated for the case when $\beta = W_2/W_1 = \frac{1}{4}$. It is seen that the maximum ordinates of these curves differ only very little from the ordinate of the points S and T so that we can state that eq. (82) gives the amplitude of the forced vibration of W_1 with a fair accuracy * provided μ is chosen in the way explained above. It remains now to show how the damping must be selected to make the resonance curves a maximum at S or at T. We begin with expression (n) by putting it into the form

$$\lambda_1{}^2/\lambda_{st}{}^2 = \frac{M\mu^2 + N}{P\mu^2 + Q},$$

* From calculations by Hahnkamm, see "Schiffbautechnische Gesellschaft, Versammlung," Nov. 1935, Berlin, it follows that the error increases with the increase in the weight of the absorber, i.e., with the increase of β. For $\beta = 0.06$ the error is 0.1 of one per cent, for $\beta = 0.7$ the error is about 1 per cent.

where M, N, P and Q are functions of γ, δ and β. Solving for μ^2 we obtain

$$\mu^2 = \frac{N - Q(\lambda_1/\lambda_{st})^2}{P(\lambda_1/\lambda_{st})^2 - M}. \tag{v}$$

As soon as the weight W_2 of the absorber has been chosen, β will be known and we obtain δ from eq. (81), γ_1^2, γ_2^2 corresponding to the points S and T, from eq. (u), and λ_1/λ_{st} from eq. (82). If all these quantities are substituted into (v) we obtain an indeterminate expression $0/0$ for μ^2, since the position of the points S and T are independent of μ. Let us take now a point very close to S on the resonance curve. If we have a maximum at S the value of λ_1/λ_{st} will not be changed by a slight shifting of the point, β and δ will also remain the same as before, and only instead of γ_1^2 we

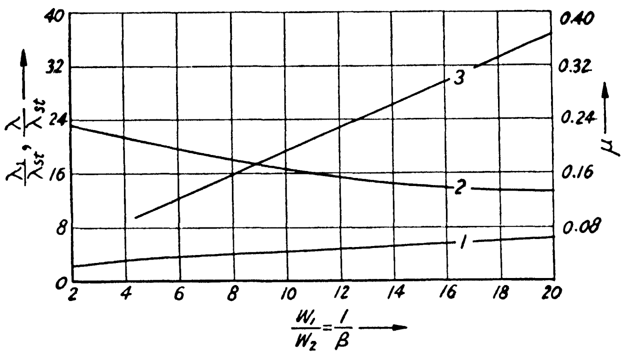

Fig. 145.

must take a slightly different quantity. With this change we shall find that the expression (v) has a definite value which is the required value of μ^2 making the tangent to the resonance curve horizontal at S. In the same manner we can get μ^2 which makes the tangent horizontal at T.

The successive steps in designing an absorber will therefore be as follows: For a given weight of the machine W_1 and its natural frequency of vibration p we choose a certain absorber weight W_2. The spring constant for the absorber is now found by the use of eq. (81); then the value of the damping follows from eq. (v). Finally the amplitude of the forced vibration is given by eq. (82). To simplify these calculations the curves in Fig. 145 can be used. As abscissas the ratios $W_1/W_2 = 1/\beta$ are taken. The ordinates of the curve 1 give the ratios λ_1/λ_{st} defining the amplitudes of vibration of the weight W_1. The curve 2 gives the amount of damping which must be used.

It remains now to design the spring of the absorber. The spring constant is determined from eq. (81). The maximum stress in the spring due to vibration may be found if we know the maximum relative displacement $\lambda = (x_2 - x_1)_{\max}$. An exact calculation of this quantity requires a complicated investigation of the motion of W_2. A satisfactory approximation can be obtained by assuming that the vibration of the system is 90 degrees behind the pulsating load $P \cos \omega t$ acting on the weight W_1. In such a case the work done per cycle is (see p. 45)

$$\pi P \lambda_1.$$

The dissipation of energy per cycle due to damping forces proportional to the relative velocity is (see p. 45)

$$\pi \alpha \omega \lambda^2.$$

Equating the energy dissipated to the work produced per cycle we obtain

$$\pi P \lambda_1 = \pi \alpha \omega \lambda^2$$

from which

$$\lambda^2 = \frac{P \lambda_1}{\alpha \omega}$$

or, by introducing our previous notations

$$\mu = \alpha g / 2W_2 p, \qquad P/k_1 = \lambda_{st}, \qquad W_2/W_1 = \beta$$

we obtain

$$\left(\frac{\lambda}{\lambda_{st}}\right)^2 = \frac{\lambda_1}{\lambda_{st}} \frac{1}{2\mu\gamma\beta}. \tag{83}$$

Since μ and β are usually small quantities the relative displacements λ, as obtained from this equation, will be several times larger than the displacement λ_1 of the weight W_1. The values of the ratio λ/λ_{st} are shown in Fig. 145 by the curve 3. Large displacements produce large stresses in the absorber spring and since these stresses are changing sign during vibration, the question of sufficient safety against future failure is of a great practical importance. The theory of the vibration absorber which has been discussed can be applied also in the case of torsional vibrations. The principal field of application of absorbers is in internal-combustion engines. The application of an absorber with Couloumb friction in the case of torsional vibrations is discussed in Art. 46.

The same principle governs also the *Schlingertank* proposed by H. Frahm in 1911* for stabilizing ships. It consists of two tanks partially filled with water, connected by two pipes (Fig. 146). The upper pipe contains an air throttle. The ship rolling in the water corresponds to the main system in Fig. 140, the impulses of the waves take the place of the disturbing force, and the water surging between the two tanks is the vibration absorber. The damping in the system is regulated by means of the air throttle. The arrangement has proved to be successful on large passenger steamers.† Another type of vibration absorber has been used by H. Frahm for eliminating vibrations in the hull of a ship. A vibratory system analogous to that of a pallograph (see Fig. 51) was attached at the stern of the ship and violent vibrations of the mass of this vibrator produced by vibration of the hull were damped out by a special hydraulic damping arrangement. It was possible in this manner to reduce to a very great extent the vibrations in the hull of the ship produced by unbalanced parts of the engine.

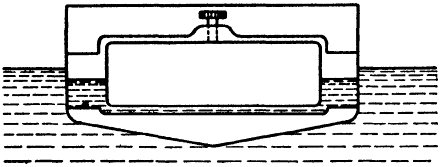

FIG. 146.

* H. Frahm, "Neuartige Schlingertanks zur Abdämpfung von Schiffsrollbewegungen," Jb. d. Schiffbautechn. Ges., Vol. 12, 1911, p. 283.

† The theory of this absorber has been discussed by M. Schuler, Proc. 2nd International Congress for Applied Mechanics, p. 219, 1926, Zürich and "Werft, Reederei, Hafen," v. 9, 1928. See also E. Hahnkamm, "Werft, Reederei, Hafen," v. 13, 1932; and Ingenieur-Archiv, v. 3, p. 251, 1932; O. Föppl, Ingenieur-Archiv, v. 5, p. 35, 1934, and Mitteilungen des Wöhler-Instituts, Heft 25, 1935; N. Minorsky, Journal of the American Society of Naval Engineers, v. 47, p. 87, 1935.

CHAPTER V

TORSIONAL AND LATERAL VIBRATION OF SHAFTS

42. Free Torsional Vibrations of Shafts.—In the previous discussion of torsional vibrations (see Art. 2) a simple problem of a shaft with two rotating masses at the ends was considered. In the following the general case of vibration of a shaft with several rotating masses will be discussed, Fig. 147. Many problems on torsional vibrations in electric machinery, Diesel engines and propeller shafts can be reduced to such a system.*

Let $I_1, I_2, I_3, \cdots$ be moments of inertia of the rotating masses about the axis of the shaft, $\varphi_1, \varphi_2, \varphi_3, \cdots$ angles of rotation of these masses during vibration, and $k_1, k_2, k_3, \cdots$ spring constants of the shaft for the length ab, bc, and cd, respectively. Then $k_1(\varphi_1 - \varphi_2), k_2(\varphi_2 - \varphi_3), \cdots$ represent torsional moments for the above lengths. If we proceed as in Art. 2 and observe that on the first disc a torque $-k_1(\varphi_1 - \varphi_2)$ acts during vibration, while on the second disc the torque is $k_1(\varphi_1 - \varphi_2) - k_2(\varphi_2 - \varphi_3)$ and so on, the differential equations of motion for consecutive discs become

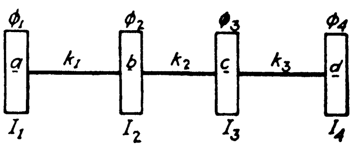

FIG. 147.

$$
\begin{aligned}
&I_1\ddot{\varphi}_1 + k_1(\varphi_1 - \varphi_2) = 0 \\
&I_2\ddot{\varphi}_2 - k_1(\varphi_1 - \varphi_2) + k_2(\varphi_2 - \varphi_3) = 0 \\
&I_3\ddot{\varphi}_3 - k_2(\varphi_2 - \varphi_3) + k_3(\varphi_3 - \varphi_4) = 0 \\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\
&I_{n-1}\ddot{\varphi}_{n-1} - k_{n-2}(\varphi_{n-2} - \varphi_{n-1}) + k_{n-1}(\varphi_{n-1} - \varphi_n) = 0 \\
&I_n\ddot{\varphi}_n - k_{n-1}(\varphi_{n-1} - \varphi_n) = 0. \qquad (a)
\end{aligned}
$$

* The bibliography of this subject can be found in the very complete investigation of torsional vibration in the Diesel engine made by F. M. Lewis; see Trans. Soc. of Naval Architects and Marine Engineers, Vol. 33, 1925, p. 109, New York. A number of practical examples are calculated in the books: W. K. Wilson, "Practical Solution of Torsional Vibration Problems," New York, 1935. W. A. Tuplin, "Torsional Vibration," New York, 1934.

Adding these equations together we get

$$I_1\ddot{\varphi}_1 + I_2\ddot{\varphi}_2 + \cdots + I_n\ddot{\varphi}_n = 0, \tag{b}$$

which means that the moment of momentum of the system about the axis of the shaft remains constant during the free vibration. In the following this moment of momentum will be taken equal to zero. In this manner any rotation of the shaft as a rigid body will be excluded and only vibratory motion due to twist of the shaft will be considered. To find the frequencies of the natural vibrations of this system we proceed as before and take the solutions of equations (a) in the form

$$\varphi_1 = \lambda_1 \cos pt, \qquad \varphi_2 = \lambda_2 \cos pt, \qquad \varphi_3 = \lambda_3 \cos pt, \cdots.$$

Substituting in equations (a) we obtain

$$\begin{aligned}
&I_1\lambda_1p^2 - k_1(\lambda_1 - \lambda_2) = 0\\
&I_2\lambda_2p^2 + k_1(\lambda_1 - \lambda_2) - k_2(\lambda_2 - \lambda_3) = 0\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&I_n\lambda_np^2 + k_{n-1}(\lambda_{n-1} - \lambda_n) = 0.
\end{aligned} \tag{c}$$

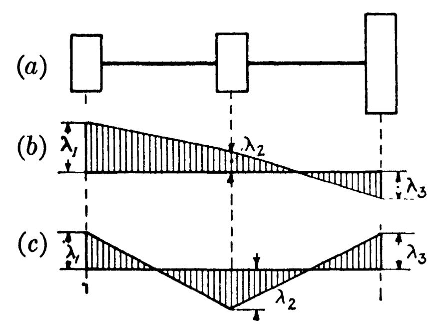

Fig. 148.

Eliminating $\lambda_1, \lambda_2, \cdots$ from these equations, we obtain an equation of the nth degree in p^2 called the *frequency equation.* The n roots of this equation give us the n frequencies corresponding to the n principal modes of vibration of the system.

The System of Three Discs.—Let us apply the above given general discussion to the problem of three discs, Fig. 148. The system of equations (c) in this case becomes:

$$\begin{aligned}
&I_1\lambda_1p^2 - k_1(\lambda_1 - \lambda_2) = 0\\
&I_2\lambda_2p^2 + k_1(\lambda_1 - \lambda_2) - k_2(\lambda_2 - \lambda_3) = 0\\
&I_3\lambda_3p^2 + k_2(\lambda_2 - \lambda_3) = 0.
\end{aligned} \tag{d}$$

From the first and the third of these equations we find that:

$$\lambda_1 = -\frac{k_1\lambda_2}{I_1p^2 - k_1}, \qquad \lambda_3 = -\frac{k_2\lambda_2}{I_3p^2 - k_2}. \tag{e}$$

Substituting these expressions into equation

$$(I_1\lambda_1 + I_2\lambda_2 + I_3\lambda_3)p^2 = 0,$$

which is obtained by adding together equations (d), we find

$$p^2\{I_1I_2I_3p^4 - [(I_1I_2+I_1I_3)k_2+(I_2I_3+I_1I_3)k_1]p^2+k_1k_2(I_1+I_2+I_3)\} = 0.$$

This is a cubic equation in p^2 of which one of the roots is $p^2 = 0$. This root corresponds to the possibility of having the shaft rotate as a rigid body without any torsion (see Art. 35). The two other roots can be readily found from the quadratic equation

$$I_1I_2I_3p^4 - [k_2(I_1I_2+I_1I_3)+k_1(I_2I_3+I_1I_3)]p^2+k_1k_2(I_1+I_2+I_3) = 0. \tag{84}$$

Let p_1^2 and p_2^2 be these two roots. Substituting p_1^2 instead of p^2 in equations (e) we find that:

$$\frac{\lambda_1}{\lambda_2} = -\frac{k_1}{I_1p_1^2 - k_1}, \quad \frac{\lambda_3}{\lambda_2} = -\frac{k_2}{I_3p_1^2 - k_2}.$$

If p_1^2 is the smaller root we shall find that one of these two ratios is positive while the other is negative; this means that during vibrations two adjacent discs will rotate in one direction while the third disc rotates in an opposite direction giving the mode of vibration shown in Fig. 148b.* For the larger root p_2^2 both ratios become negative and the mode of vibration, corresponding to the higher frequency is shown in Fig. 148c. During this vibration the middle disc rotates in the direction opposite to the rotation of the two other discs.

The Case of Many Discs.—In the case of four discs we shall have four equations in the system (c), and proceeding as in the previous case we get a frequency equation of fourth degree in p^2. One of the roots is again zero so that for calculating the remaining three roots we obtain a cubic equation. To simplify the writing of this equation, let us introduce the notations

$$\frac{k_1}{I_1} = \alpha_1, \quad \frac{k_1}{I_2} = \alpha_2, \quad \frac{k_2}{I_2} = \alpha_3, \quad \frac{k_2}{I_3} = \alpha_4, \quad \frac{k_3}{I_3} = \alpha_5, \quad \frac{k_3}{I_4} = \alpha_6$$

$$\alpha_1 + \alpha_2 = a_1, \quad \alpha_3 + \alpha_4 = a_2, \quad \alpha_5 + \alpha_6 = a_3.$$

Then the frequency equation is

$$p^6 - p^4(a_1 + a_2 + a_3) + p^2(a_1a_2 + a_1a_3 + a_2a_3 - \alpha_2\alpha_3 - \alpha_4\alpha_5) - (a_1a_2a_3 - a_2\alpha_2\alpha_3 - a_1\alpha_4\alpha_5) = 0. \tag{85}$$

* It is assumed that $I_3/k_2 > I_1/k_1$.

In solving this equation one of the approximate methods for calculating the roots of algebraic equations of higher degree must be used.*

When the number of discs is larger than four the derivation of the frequency equation and its solution become too complicated and the calculation of frequencies is usually made by one of the approximate methods.

Geared Systems.—Sometimes we have to deal with geared systems as shown in Fig. 149*a*, instead of with a single shaft. The general equations

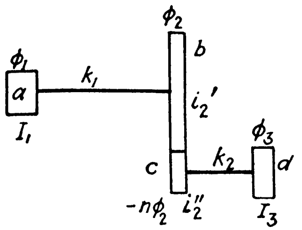

Fig. 149*a*.

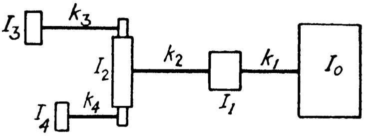

Fig. 149*b*.

of vibration of such systems can be readily derived. Considering the system in Fig. 149*a*, let

I_1, I_3 be moments of inertia of rotating masses.
φ_1, φ_3 are the corresponding angles of rotation.
i_2', i_2'' are moments of inertia of gears.
n is gear ratio.
φ_2, $-n\varphi_2$ are angles of rotation of gears.
k_1, k_2 are spring constants of shafts.

Then the kinetic energy of the system will be

$$T = \frac{I_1\dot{\varphi}_1^2}{2} + \frac{i_2'\dot{\varphi}_2^2}{2} + \frac{i_2''(n\dot{\varphi}_2)^2}{2} + \frac{I_3\dot{\varphi}_3^2}{2}. \tag{f}$$

The potential energy of the system is

$$V = \tfrac{1}{2}k_1(\varphi_1 - \varphi_2)^2 + \tfrac{1}{2}k_2(\varphi_3 + n\varphi_2)^2. \tag{g}$$

Letting

$$i_2' + n^2 i_2'' = I_2; \qquad n^2 I_3 = I_3',$$

$$\varphi_3 = -n\varphi_3'; \qquad k_2 n^2 = k_2'. \tag{h}$$

* Such methods are discussed in v. Sanden's book, "Practical Analysis."

The equations (f) and (g) become

$$T = \frac{I_1\dot{\varphi}_1^2}{2} + \frac{I_2\dot{\varphi}_2^2}{2} + \frac{I_3'(\dot{\varphi}_3')^2}{2},$$

$$V = \tfrac{1}{2}k_1(\varphi_1 - \varphi_2)^2 + \tfrac{1}{2}k_2'(\varphi_2 - \varphi_3')^2.$$

These expressions have the same form as the expressions for T and V which can be written for a single shaft. It can be concluded from this that the differential equations of vibration of the geared system shown in Fig. 149 will be the same as those of a single shaft with discs provided the notations shown in eqs. (h) are used. This conclusion can be expanded also to the case of a geared system with more than two shafts.*

Another arrangement of a geared system is shown in Fig. 149b, in which I_0, I_1, I_2, $\cdots$ are the moments of inertia of the rotating masses; k_1, k_2, $\cdots$ torsional rigidities of the shafts. Let, n be gear ratio, φ_0, φ_1, φ_2, $\cdots$ — angles of rotation of discs I_0, I_1, I_2 $\cdots$. If I_0 is very large in comparison with the other moments of inertia we can take $\varphi_0 = 0$, then the kinetic and the potential energy of the system will be

$$\begin{aligned} T &= \tfrac{1}{2}(I_1\dot{\varphi}_1^2 + I_2\dot{\varphi}_2^2 + I_3\dot{\varphi}_3^2 + I_4\dot{\varphi}_4^2), \\ V &= \tfrac{1}{2}\left\{k_1\varphi_1^2 + k_2(\varphi_2 - \varphi_1)^2 + k_3(\varphi_3 + n\varphi_2)^2 + k_4(\varphi_4 + n\varphi_2)^2\right\}, \end{aligned}$$

and Lagrange's differential equations of motion become

$$\begin{aligned} I_1\ddot{\varphi}_1 + k_1\varphi_1 - k_2(\varphi_2 - \varphi_1) &= 0, \\ I_2\ddot{\varphi}_2 + k_2(\varphi_2 - \varphi_1) + nk_3(\varphi_3 + n\varphi_2) + nk_4(\varphi_4 + n\varphi_2) &= 0, \\ I_3\ddot{\varphi}_3 + k_3(\varphi_3 + n\varphi_2) &= 0, \\ I_4\ddot{\varphi}_4 + k_4(\varphi_4 + n\varphi_2) &= 0, \end{aligned}$$

from which the frequency equation can be obtained in the same manner as before and the frequencies will then be represented by the roots of this equation.

PROBLEMS

Determine the natural frequencies of a steel shaft with three discs, Fig. 148, if the weights of the discs are 3000 lb., 2000 lb. and 1000 lb., the diameters of the discs are 40 in., the distances between the discs are $l_1 = l_2 = 30$ in., the diameter of the shaft is 5 in. and the modulus of elasticity in shear is $G = 11.5 \cdot 10^6$ lb. per sq. in. Determine the ratios between the angular deflections $\lambda_1 : \lambda_2$, $\lambda_2 : \lambda_3$ for the two principal modes of vibration.

* Such systems are considered in the paper by T. H. Smith, "Nodal Arrangements of Geared Drives," Engineering, 1922, pp. 438 and 467.

Solution. Making our calculations in inches and in pounds, we find that:

$$I_1 = 1553, \quad I_2 = 1035, \quad I_3 = 517.7, \quad k_1 = k_2 = 23.5 \cdot 10^6.$$

Eq. (84) becomes

$$p^4 - 106000\, p^2 + 2060.10^6 = 0,$$

from which

$$p_1^2 = 25600, \quad p_2^2 = 80400.$$

The corresponding frequencies are:

$$f_1 = \frac{p_1}{2\pi} = 25.5 \text{ per sec.} \quad f_2 = \frac{p_2}{2\pi} = 45.2 \text{ per sec.}$$

The ratios of amplitudes for the fundamental mode of vibration are:

$$\lambda_1/\lambda_2 = -1.44, \quad \lambda_3/\lambda_2 = 2.29.$$

For the higher mode of vibration

$$\lambda_1/\lambda_2 = -0.232, \quad \lambda_3/\lambda_2 = -1.30.$$

43. Approximate Methods of Calculating Frequencies of Natural Vibrations.—In practical applications it is usually the lowest frequency or the two lowest frequencies of vibration of a shaft with several discs that are important and in many cases these can be approximately calculated by using the results obtained in the case of two and three discs. Take as a first example a shaft with four discs of which the moments of inertia are $I_1 = 302$ lb. in. sec.^{2}, $I_2 = 87{,}500$ lb. in. sec.^{2}, $I_3 = 1200$ lb. in. sec.^{2}, $I_4 = 0.373$ lb. in. sec.^{2}. The spring constants of the three portions of the shaft are $k_1 = 316 \cdot 10^6$ lb. in. per radian, $k_2 = 114.5 \cdot 10^6$ lb. in. per radian, $k_3 = 1.09 \cdot 10^6$ lb. in. per radian. Since I_1 and I_4 are very small we can neglect them entirely in calculating the lowest frequency and consider only the two discs I_2 and I_3. Applying equation (17) for this system, we obtain

$$f_1 = \frac{1}{2\pi}\sqrt{\frac{(I_2 + I_3)k_2}{I_2 I_3}} = 49.6 \text{ per sec.}$$

In dealing with the vibration of the disc I_1 we can consider the disc I_2 as being infinitely large and assume that it does not vibrate, then the frequency of the disc I_1, from eq. (14), is

$$f_2 = \frac{1}{2\pi}\sqrt{\frac{k_1}{I_1}} = 163 \text{ per sec.}$$

Noting again that the disc I_4 is very small in comparison with I_3 and neglecting the motion of the latter disc we find

$$f_3 = \frac{1}{2\pi}\sqrt{\frac{k_3}{I_4}} = 272 \text{ per sec.}$$

A more elaborate calculation for this case by using the cubic equation (84) gives $f_1 = 49.5$, $f_2 = 163$, $f_3 = 272$, so that for the given proportions of the discs it is not necessary to go into a refined calculation.

As a second example let us consider the system shown in Fig. 150, where the moments of inertia of the generator, flywheel, of six cylinders and two air pumps, and also the distances between these masses are given.*

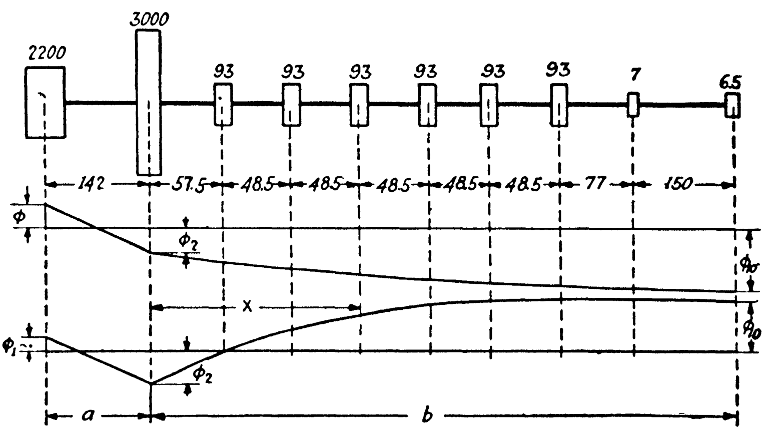

FIG. 150.

The shaft is replaced by an *equivalent shaft* of uniform section (see p. 271) with a torsional rigidity $C = 10^{10}$ kg. $\times$ cm.^{2} Due to the fact that the masses of the generator and of the flywheel are much larger than the remaining masses a good approximation for the frequency of the lowest type of vibration can be obtained by replacing all the small masses by one mass having a moment of inertia, $I_3 = 93 \times 6 + 7 + 6.5 \approx 572$ and located at the distance $57.5 + 2.5 \times 48.5 \approx 179$ centimeters from the flywheel. Reducing in this manner the given system to three masses only the frequencies can be easily calculated from eq. (84) and we obtain $p_1{}^2 = 49{,}000$ and $p_2{}^2 = 123{,}000$. The exact solution for the same problem gives $p_1{}^2 =$

* This example is discussed in the book by Holzer mentioned below (see p. 263). Kilogram and centimeter are taken as units.

49,840 and $p_2^2 = 141{,}000$. It is seen that a good approximation is obtained for the fundamental type of vibration. In order to get a still better approximation Rayleigh's method can be used (see Art. 16).

Rayleigh's Method.—Let a and b denote the distances of the flywheel from the ends of the shaft and assume that the shapes of the two principal modes of vibrations are such as shown in Fig. 150 (*b*) and (*c*) and that the part b of the deflection curve can be replaced by a parabola so that the angle of twist φ for any section distant x from the flywheel is given by the equation

$$\varphi = \varphi_2 + \frac{(\varphi_{10} - \varphi_2)(2b - x)x}{b^2}. \tag{a}$$

It is easy to see that for $x = 0$ and $x = b$ the angle φ in the above equation assumes the values φ_2 and φ_{10} respectively. By eq. (*a*) and the first of eqs. (*c*) Art. 42 we have:

$$\varphi_2 = \varphi_1\left(1 - \frac{I_1p^2}{k_1}\right). \tag{b}$$

The angles of rotation of all other masses can be represented as functions of φ_1 and φ_{10} and these latter two angles can be considered as the generalized coordinates of the given system.

Then the potential energy of the system is

$$V = \frac{(\varphi_1 - \varphi_2)^2C}{2a} + \frac{1}{2}C\int_0^b\left(\frac{d\varphi}{dx}\right)^2dx = \frac{C}{2}\left\{\frac{\varphi_1^2\gamma^2}{a} + \frac{4}{3}\frac{\{\varphi_{10} - \varphi_1(1 - \gamma)\}^2}{b}\right\}, \tag{c}$$

in which

$$\gamma = \frac{I_1p^2}{k_1}, \text{ and } C \text{ is torsional rigidity of the shaft.} \tag{d}$$

The kinetic energy of the system will be

$$T = \frac{1}{2}\sum_{k=1}^{k=10}\frac{I_k\dot{\varphi}_k^2}{2},$$

or by using the eqs. (*a*) and (*b*) and letting x_k = the distance from the flywheel to any rotating mass k and $\alpha_k = \dfrac{(2b - x_k)x_k}{b^2}$

$$T = \frac{1}{2}I_1\dot{\varphi}_1^2 + \frac{1}{2}\sum_{k=2}^{k=10} I_k\{\dot{\varphi}_{10}\alpha_k + \dot{\varphi}_1(1 - \gamma)(1 - \alpha_k)\}^2. \tag{e}$$

Substituting (*e*) and (*c*) in Lagrange's equations (73) and putting, as before,

$$\varphi_1 = \lambda_1 \cos(pt + \beta); \qquad \varphi_{10} = \lambda_{10}\cos(pt + \beta),$$

the following two equations will be obtained:

$$\lambda_1 \left\{ \gamma - \frac{4}{3}\frac{a}{b}(1-\gamma) + \frac{\gamma(1-\gamma)}{I_1}\sum_{k=2}^{k=10} I_k(1-\alpha_k)^2 \right\} + \lambda_{10}\left\{ \frac{4}{3}\frac{a}{b} + \frac{\gamma}{I_1}\sum_{k=2}^{k=10} I_k\alpha_k(1-\alpha_k) \right\} = 0,$$

$$\lambda_1 \left\{ \frac{4}{3}\frac{a}{b}(1-\gamma) + \frac{\gamma(1-\gamma)}{I_1}\sum_{k=2}^{k=10} I_k\alpha_k(1-\alpha_k) \right\} + \lambda_{10}\left\{ -\frac{4}{3}\frac{a}{b} + \frac{\gamma}{I_1}\sum_{k=2}^{k=10} I_k\alpha_k^2 \right\} = 0.$$

By equating the determinant of these equations to zero the frequency equation will be obtained, the two roots of which will give us the frequencies of the two modes of vibration shown in Fig. 150. All necessary calculations are given in the table on p. 262. Then from the frequency equation the smaller root will be

$$\gamma = 1.563,$$

and from (d) we obtain

$$p^2 = 50000.$$

The error of this approximate solution as compared with the exact solution given above is only $\frac{1}{6}\%$.

The second root of the frequency equation gives the frequency of the second mode of vibration with an accuracy of 4.5%. It should be noted that in using this approximate method the effect of the mass of the shaft on the frequency of the system can easily be calculated.*

As soon as we have an approximate value of a frequency, we can improve the accuracy of the solution by the method of successive approximations. For this purpose the equations (c) (p. 254), must be written in the form:

$$\lambda_2 = \lambda_1 - \frac{I_1 p^2}{k_1}\lambda_1, \tag{f}$$

$$\lambda_3 = \lambda_2 - \frac{p^2}{k_2}(I_1\lambda_1 + I_2\lambda_2), \tag{g}$$

$$\lambda_4 = \lambda_3 - \frac{p^2}{k_3}(I_1\lambda_1 + I_2\lambda_2 + I_3\lambda_3), \tag{h}$$

. . . .

. . . .

Making now a rough estimate of the value p^2 and taking an arbitrary value for λ_1, the angular deflection of the first disc, the corresponding value of λ_2 will be found from eq. (f). Then, from eq. (g) λ_3 will be

* See writer's paper in the Bulletin of the Polytechnical Institute in S. Petersburg (1905).

k	I_k	x_k	$\frac{x_k}{b}$	$2 - \frac{x_k}{b}$	α_k	$1 - \alpha_k$	α_k^2	$I_k\alpha_k^2$	$(1-\alpha_k)^2$	$I_k(1-\alpha_k)^2$	$\alpha_k(1-\alpha_k)$	$I_k\alpha_k(1-\alpha_k)$
1	2200	—	—	—	—	—	—	—	—	—	—	—
2	3000	0	0	2	0	1	0	0	1	3000	0	0
3	93	57.5	.1091	1.8909	.2062	7938	.0425		6301		.1637	
4	93	106	.2011	1.7989	.3618	.6382	.1309		.4073		.2309	
5	93	154.5	.2932	1.7068	5004	.4996	.2504		.2496		.2500	
6	93	203	.3852	1.6148	.6220	.3780	.3869	186 2	.1429	133.8	.2351	121.2
7	93	251.5	.4772	1.5228	.7268	.2732	.5282		.0746		.1986	
8	93	300	.5693	1.4307	.8145	.1855	.6634		0344		.1511	
9	7	377	.7154	1.2846	.9190	.0810	.8446	5 91	0065		.0744	.52
10	6 5	527	1.0000	1 0000	1.0000	0	1.0000	6 50	0	0	0	0

$$\sum_{k=2}^{k=10} I_k\alpha_k^2 = 198.6; \sum_{k=2}^{k=10} I_k(1-\alpha_k)^2 = 3133\ 8; \sum_{k=2}^{k=10} I_k\alpha_k(1-\alpha_k) = 121.7.$$

found; λ_4 from eq. (h) and so on. If the magnitude of p^2 had been chosen correctly, the equation

$$I_1\lambda_1p^2 + I_2\lambda_2p^2 + \cdot \cdot \cdot I_n\lambda_np^2 = 0,$$

representing the sum of the eqs. (c) (p. 254), would be satisfied. Otherwise the angles λ_2, λ_3, $\cdot \cdot \cdot$ would have to be calculated again with a new estimate for p^2.* It is convenient to put the results of these calculations in tabular form. As an example, the calculations for a Diesel installation, shown in Fig. 151, are given in the tables on p. 264.†

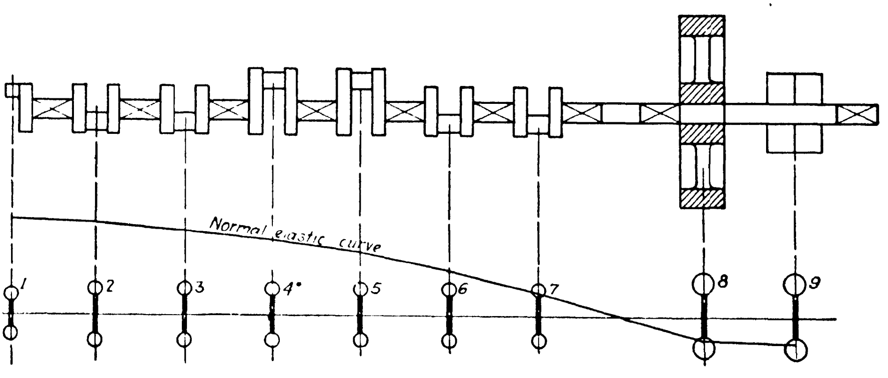

FIG. 151.

Column 1 of the tables gives the moments of inertia of the masses, inch, pound and second being taken as units. Column 3 begins with an arbitrary value of the angle of rotation of the first mass. This angle is taken equal to 1. Column 4 gives the moments of the inertia forces of the consecutive masses and column 5 the total torque of the inertia forces of all masses to the left of the cross section considered. Dividing the torque by the spring constants given in column 6, we obtain the angles of twist for consecutive portions of the shaft. These are given in column 7. The last number in column 5 represents the sum of the moments of the inertia forces of all the masses. This sum must be equal to zero in the case of free vibration. By taking $p = 96.2$ in the first table, the last value in column 5 becomes positive. For $p = 96.8$, taken in the second table, the corresponding value is negative. This shows that the exact value of

* Several examples of this calculation may be found in the book by H. Holzer, "Die Berechnung der Drehschwingungen," 1921, Berlin, J. Springer. See also F. M., Lewis, loc. cit., and Max Tolle, "Regelung der Kraftmaschinen," 3d Ed., 1921.

† These calculations were taken from the paper of F. M. Lewis, mentioned above.

Table for p = 96,2; p^2 = 9250

Mass No.	1	2	3	4	5	6	7
	I	Ip^2	λ	$Ip^2\lambda$	$\Sigma Ip^2\lambda$	k	$\frac{1}{k}\Sigma Ip^2\lambda$
1	708	$6{,}55\times10^6$	1	$6{,}55\times10^6$	$6{,}55\times10^6$	2070×10^6	.0031
2	3920	$36{,}25\times10^6$	.9969	$36{,}1\times10^6$	$42{,}65\times10^6$	730×10^6	.0585
3	3920	$36{,}25\times10^6$	.9383	$34{,}0\times10^6$	$76{,}65\times10^6$	730×10^6	.1050
4	3920	$36{,}25\times10^6$	.8333	$30{,}2\times10^6$	$106{,}85\times10^6$	730×10^6	.1462
5	3920	$36{,}25\times10^6$	.6871	$24{,}9\times10^6$	$131{,}75\times10^6$	730×10^6	.1803
6	3920	$36{,}25\times10^6$	.5068	$18{,}38\times10^6$	$150{,}1\times10^6$	730×10^6	.2060
7	3920	$36{,}25\times10^6$	.3008	$10{,}90\times10^6$	$161{,}0\times10^6$	402×10^6	.4010
8	139800	1293×10^6	−.1002	$-130{,}0\times10^6$	$31{,}0\times10^6$	1334×10^6	0233
9	26400	244×10^6	−.1235	$-\ 30{,}2\times10^6$	+800,000		

Table for p = 96,8; p^2 = 9380

Mass No.	1	2	3	4	5	6	7
	I	Ip^2	λ	$Ip^2\lambda$	$\Sigma Ip^2\lambda$	k	$\frac{1}{k}\Sigma Ip^2\lambda$
1	708	$6{,}65\times10^6$	1	$6{,}65\times10^6$	$6{,}65\times10^6$	2070×10^6	.0032
2	3920	$36{,}8\times10^6$	.9968	$36{,}7\times10^6$	$43{,}35\times10^6$	730×10^6	.0594
3	3920	$36{,}8\times10^6$	.9374	$34{,}5\times10^6$	$77{,}85\times10^6$	730×10^6	.1069
4	3920	$36{,}8\times10^6$	.8305	$30{,}6\times10^6$	$108{,}45\times10^6$	730×10^6	.1487
5	3920	$36{,}8\times10^6$	.6818	$25{,}1\times10^6$	$133{,}55\times10^6$	730×10^6	.1830
6	3920	$36{,}8\times10^6$	.4988	$18{,}36\times10^6$	$151{,}91\times10^6$	730×10^6	2080
7	3920	$36{,}8\times10^6$	.2908	$10{,}70\times10^6$	$162{,}61\times10^6$	402×10^6	.4040
8	139800	1312×10^6	−.1132	$-148{,}5\times10^6$	$14{,}11\times10^6$	1334×10^6	.0106
9	26400	248×10^6	−.1238	$-30{,}70\times10^6$	$-16{,}59\times10^6$		

p lies between the above two values and the correct values in columns 3 and 5 will be obtained by interpolation. By using the values in column 3, the elastic curve representingthe *mode of vibration* can be constructed as shown in Fig. 151. Column 5 gives the corresponding torque for each portion of the shaft when the amplitude of the first mass is 1 radian. If this amplitude has any other value λ_1, the amplitudes and the torque of the other masses may be obtained by multiplying the values in columns 3 and 5 by λ_1.

44. Forced Torsional Vibration of a Shaft with Several Discs.—If a torque $M_t \sin \omega t$ is applied to one of the discs forced vibrations of the period $\tau = 2\pi/\omega$ will be produced; moreover the vibration of each disc will be of the form $\lambda \sin \omega t$. The procedure of calculating the amplitudes of forced vibration will now be illustrated by an example.

Let us take a shaft (Fig. 152) with four discs of which the moments of inertia are $I_1 = 777$, $I_2 = 518$, $I_3 = I_4 = 130$, and the spring constants are $k_1 = 24.6 \cdot 10^6$, $k_2 = k_3 = 36.8 \cdot 10^6$, inches, pounds and seconds being

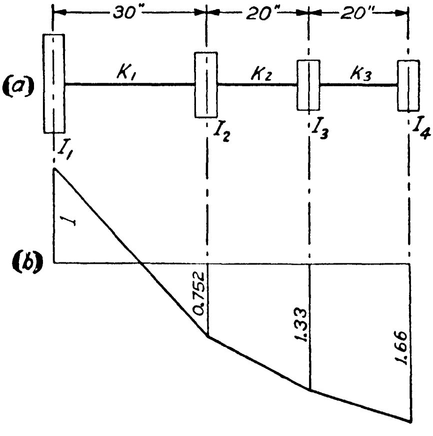

FIG. 152.

taken as units. Assume that a pulsating torque $M_t \sin \omega t$ is acting on the first disc and that it is required to find the amplitudes of the forced vibration of all the discs for the given frequency $\omega = \sqrt{31{,}150}$. The equations of motion in this case are

$$\begin{aligned}
&I_1\ddot{\varphi}_1 + k_1(\varphi_1 - \varphi_2) = M_t \sin \omega t\\
&I_2\ddot{\varphi}_2 - k_1(\varphi_1 - \varphi_2) + k_2(\varphi_2 - \varphi_3) = 0\\
&I_3\ddot{\varphi}_3 - k_2(\varphi_2 - \varphi_3) + k_3(\varphi_3 - \varphi_4) = 0\\
&I_4\ddot{\varphi}_4 - k_3(\varphi_3 - \varphi_4) = 0.
\end{aligned} \tag{a}$$

Substituting in these equations

$$\varphi_1 = \lambda_1 \sin \omega t, \quad \varphi_2 = \lambda_2 \sin \omega t, \quad \cdots$$

we obtain

$$\begin{aligned}
&I_1\lambda_1\omega^2 - k_1(\lambda_1 - \lambda_2) = -M_t\\
&I_2\lambda_2\omega^2 + k_1(\lambda_1 - \lambda_2) - k_2(\lambda_2 - \lambda_3) = 0\\
&I_3\lambda_3\omega^2 + k_2(\lambda_2 - \lambda_3) - k_3(\lambda_3 - \lambda_4) = 0\\
&I_4\lambda_4\omega^2 + k_3(\lambda_3 - \lambda_4) = 0.
\end{aligned} \tag{b}$$

By adding these equations we find that

$$\omega^2(I_1\lambda_1 + I_2\lambda_2 + I_3\lambda_3 + I_4\lambda_4) = - M_t. \qquad (c)$$

If λ_1 is the amplitude of the first disc the amplitude of the second disc is found from the first of equations (*b*).

$$\lambda_2 = \lambda_1 - \frac{I_1\lambda_1}{k_1}\omega^2 - \frac{M_t}{k_1}. \qquad (d)$$

Substituting this expression into the second of equations (*b*) we find λ_3 and from the third of equations (*b*) we find λ_4. Thus all the amplitudes will be expressed by λ_1. Substituting them in equation (*c*), we obtain a linear equation in λ_1.

It is advantageous to make all the calculations in tabular form as shown in the table below:

1	2	3	4	5
I	λ	$I\omega^2\lambda$	$\Sigma I\omega^2\lambda$	$\frac{1}{k}\Sigma I\omega^2\lambda$
777	λ_1	$24\ 2\cdot 10^6\lambda_1$	$24\ 2\cdot 10^6\lambda_1$	$0{,}984\lambda_1$
518	$0.016\lambda_1 - 4\ 07\cdot 10^{-8}M_t$	$.256\cdot 10^6\lambda_1 - 0\ 655M_t$	$24.5\cdot 10^6\lambda_1 - 0.655M_t$	$0\ 666\lambda_1 - 1\ 78\cdot 10^{-8}M_t$
130	$-0.650\lambda_1 - 5.01\cdot 10^{-8}M_t$	$-2.63\cdot 10^6\lambda_1 - 0\ 203M_t$	$21.9\cdot 10^6\lambda_1 - 0.858M_t$	$0.595\lambda_1 - 2.33\cdot 10^{-8}M_t$
130	$-1.24\lambda_1\ - 5\ 41\cdot 10^{-8}M_t$	$-5\ 02\cdot 10^6\lambda_1 - 0.219M_t$	$16.9\cdot 10^6\lambda_1 - 1.077M_t$	

We begin with the first row of the table. By using the given numerical values of I_1, ω^2 and k_1 we calculate $I_1\omega^2$ and $I_1\omega^2/k_1$. Starting with the second row we calculate λ_2 by using eq. (*d*) and the figures from the first row. In this way the expression in the second column and the second row is obtained. Multiplying it with $\omega^2 I_2$ the expression in the third column and the second row is obtained. Adding it to the expression in the fourth column of the first row and dividing afterwards by k_2 the last two terms of the second row are obtained. Having these quantities, we start with the third row by using the second of equations (*b*) for calculating λ_3 and then continue our calculations as before. Finally we obtain the expression in the fourth row and the fourth column which represents the left side of equation (*c*). Substituting this expression into equation (*c*), we find the equation for calculating λ_1

$$16.9\cdot 10^6\lambda_1 - 1.077M_t = - M_t.$$

This gives

$$\lambda_1 = \frac{0.077\,M_t}{16.9 \cdot 10^6}.$$

If this value of λ_1 be substituted into the expressions of the second column, the amplitudes of the forced vibration of all the discs may be calculated. Having these amplitudes we may calculate the angles of twist of the shaft between the consecutive discs since they are equal to $\lambda_1 - \lambda_2$, $\lambda_2 - \lambda_3$ and $\lambda_3 - \lambda_4$. With these values of the angles of twist and with the known dimensions of the shaft the shearing stresses produced by the forced vibration may be found by applying the known formula of strength of material.

Effect of Damping on Torsional Vibrations at Resonance.—If the period of the external harmonic torque coincides with the period of one of the natural modes of vibration of the system, a condition of resonance takes place. This mode of vibration becomes very pronounced and the damping forces must be taken into consideration in order to obtain the actual value of the amplitude of vibration.* Assuming that the damping force is proportional to the velocity and neglecting the effect of this force on the *mode of vibration*, i.e., assuming that the ratios between the amplitudes of the steady forced vibration of the rotating masses are the same as for the corresponding type of free vibration, the approximate values of the amplitudes of forced vibration may be calculated as follows: Let $\varphi_m = \lambda_m \sin pt$ be the angle of rotation of the m^{th} disc during vibration on which damping is acting. Then the resisting moment of the damping forces will be

$$-c\frac{d\varphi_m}{dt} = -c\lambda_m p \cos pt,$$

where c is a constant depending upon the damping condition. The phase difference between the torque which produces the forced vibration and the displacement must be 90 degrees for resonance. Hence we take this moment in the form $M_t \cos pt$. Assuming $\varphi_n = \lambda_n \sin pt$ for the angle of rotation of the n^{th} mass on which the torque is acting, the amplitude of the forced vibration will be found from the condition that in the steady state of forced vibration the work done by the harmonic torque during one

* The approximate method of calculating forced vibration with damping has been developed by H. Wydler in the book: "Drehschwingungen in Kolbenmaschinenanlagen." Berlin, 1922. See also F. M. Lewis, loc. cit., p. 253; John F. Fox, Some Experiences with Torsional Vibration Problems in Diesel Engine Installations, Journal Amer. Soc. of Naval Engineers 1926, and G. G. Eichelberg, "Torsionsschwingungausschlag," Stodola Festschrift, Zürich, 1929, p. 122.

oscillation must be equal to the energy absorbed at the damping point. In this manner we obtain

$$\int_0^{\frac{2\pi}{p}} c\frac{d\varphi_m}{dt}\frac{d\varphi_m}{dt}dt = \int_0^{\frac{2\pi}{p}} M_t \cos pt\frac{d\varphi_n}{dt}dt,$$

or substituting

$$\varphi_m = \lambda_m \sin pt; \qquad \varphi_n = \lambda_n \sin pt,$$

we obtain

$$\lambda_m = \frac{M_t}{cp}\frac{\lambda_n}{\lambda_m}, \tag{e}$$

and the amplitude of vibration for the first mass will be

$$\lambda_1 = \frac{M_t}{cp}\frac{\lambda_n}{\lambda_m}\frac{\lambda_1}{\lambda_m}. \tag{f}$$

Knowing the damping constant c and taking the ratios λ_n/λ_m and λ_1/λ_m from the *normal elastic curve* (see Fig. 151) the amplitudes of forced vibration may be calculated for the case of a simple harmonic torque with damping applied at a certain section of the shaft.

Consider again the example of the four discs shown in Fig. 152. By using the method of successive approximation we shall find with sufficient accuracy that the circular frequency of the lowest mode of vibration is approximately $p = 235$ radians per second, and that the ratios of the amplitudes for this mode of vibration are $\lambda_2/\lambda_1 = -0.752$, $\lambda_3/\lambda_1 = -1.33$, $\lambda_4/\lambda_1 = -1.66$. The corresponding normal elastic curve is shown in Fig. 152*b*. Assume now that the periodic torque $M \cos pt$ is applied at the first disc and that the damping is applied at the fourth disc.* Then from equation (f)

$$\lambda_1 = \frac{M}{cp}\frac{\lambda_1}{\lambda_4}\frac{\lambda_1}{\lambda_4}\cdot$$

Substituting the value from the normal elastic curve for the ratio λ_1/λ_4 we find

$$\lambda_1 = 0.36\frac{M_t}{cp}\cdot$$

From this equation the amplitude λ_1 can be calculated for any given torque M_t and any given value of damping factor c.

* The same reasoning holds if damping is applied to any other disc.

If several simple harmonic torques are acting on the shaft, the resultant amplitude λ_1, of the first mass, may be obtained from the equation (f) above by the principle of superposition. It will be equal to

$$\lambda_1 = \frac{1}{cp}\frac{\lambda_1}{\lambda_m{}^2}\Sigma M_t \lambda_n, \tag{g}$$

where the summation sign indicates the vector sum, each torque being taken with the corresponding phase.

In actual cases the external torque is usually of a more complicated nature. In the case of a Diesel engine, for instance, the *turning effort* produced by a single cylinder depends on the position of the crank, on the gas pressure and on inertia forces. The *turning effort curve* of each cylinder may be constructed from the corresponding gas pressure diagram, taking into account the inertia forces of the reciprocating masses. In analyzing forced vibrations this curve must be represented by a trigonometrical series *

$$f(\varphi) = a_0 + a_1 \cos \varphi + a_2 \cos 2\varphi + \cdots \\ + b_1 \sin \varphi + b_2 \sin 2\varphi + \cdots, \tag{d}$$

in which $\varphi = 2\pi$ represents the period of the curve. This period is equal to one revolution of the crankshaft in a two-cycle engine and to two revolutions in a four-cycle engine. The condition of resonance occurs and a critical speed will be obtained each time when the frequency of one of the terms of the series (d) coincides with the frequency of one of the natural modes of vibration of the shaft. For a single cylinder in a two-cycle engine there will be obtained in this manner critical speeds of the order 1, 2, 3, . . ., where the index indicates the number of vibration cycles per revolution of the crankshaft. In the case of a four-cycle engine, we may have critical speeds of the order ½, 1, 1½, . . .; i.e., of every integral order and half order. There will be a succession of such critical speeds for each mode of natural vibration. The amplitude of a forced vibration of a given type produced by a single cylinder may be calculated as has been explained before. In order to obtain the summarized effect of all cylinders, it will be necessary to use the principle of superposition, taking the turning effort of each cylinder at the corresponding phase. In particular cases, when the number of vibrations per revolution is equal to, or a multiple of the number of firing impulses (a major critical speed) the phase difference is zero and

* Examples of such an analysis may be found in the papers by H. Wydler, loc. cit., p. 152, and F. M. Lewis, loc. cit., p. 253.

the vibrations produced by the separate cylinders will be simply added together. Several examples of the calculation of amplitudes of forced vibration are to be found in the papers by H. Wydler and F. M. Lewis mentioned above. They contain also data on the amount of damping in such parts as the marine propeller, the generator, and the cylinders as well as data on the losses due to internal friction.* The application in particular cases of the described approximate method gives satisfactory accuracy in computing the amplitude of forced vibration and the corresponding maximum stress.

45. Torsional Vibration of Diesel Engine Crankshafts.—We have been dealing so far with a uniform shaft having rigid discs mounted on it.

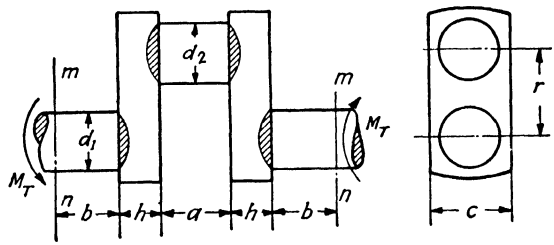

FIG. 153.

There are, however, cases in which the problem of torsional vibration is more complicated. An example of such a problem we have in the torsional vibrations of Diesel-Engine crankshafts. Instead of a cylindrical shaft we have here a crankshaft of a complicated form and instead of rotating circular dics we have rotating cranks connected to reciprocating masses of the engine. If the crankshaft be replaced by an equivalent cylindrical shaft the torsional rigidity of one crank (Fig. 153) must be considered first. This rigidity depends on the conditions of constraint at the bearings. Assuming that the clearances in the bearings are such that free displacements of the cross sections m–n and m–n during twist are possible, the angle of twist produced by a torque M_t can be easily obtained. This angle consists of three parts: (*a*) twist of the journals, (*b*) twist of the crankpin and (*c*) bending of the web.

* Bibliography on this subject and some new data on internal friction may be found in the book by E. Lehr, "Die Abkürzungsverfahren zur Ermittelung der Schwingungsfestigkeit," Stuttgart, dissertation, 1925. See also E. Jaquet, "Stodola's Festschrift," p. 308; S. F. Dorey, Proc. I. Mech. E. v. 123, p. 479, 1932; O. Föppl, The Iron and Steel Institute, October, 1936.

Let $C_1 = \dfrac{\pi d_1^4 G}{32}$ be the torsional rigidity of the journal,

$C_2 = \dfrac{\pi d_2^4 G}{32}$ be the torsional rigidity of the crankpin,

$B = \dfrac{hc^3}{12} E$ be the flexural rigidity of the web.

In order to take into account local deformations of the web in the regions shaded in the figure, due to twist, the lengths of the journal and of the pin are taken equal to $2b_1 = 2b + .9h$ and $a_1 = a + .9h$, respectively.* The angle of twist θ of the crank produced by a torque M_t will then be

$$\theta = \frac{2b_1 M_t}{C_1} + \frac{a_1 M_t}{C_2} + \frac{2r M_t}{B}.$$

In calculating the torsional vibrations of a crankshaft every crank must be replaced by an equivalent shaft of uniform cross section of a torsional rigidity C. The length of the equivalent shaft will be found from

$$\frac{M_t l}{C} = \theta,$$

in which θ is the angle of twist calculated above.

Then the length of equivalent shaft will be,

$$l = C\left(\frac{2b_1}{C_1} + \frac{a_1}{C_2} + \frac{2r}{B}\right). \tag{86}$$

Another extreme case will be obtained on the assumption that the constraint at the bearings is complete, corresponding to no clearances. In this case the length l of the *equivalent shaft* will be found from the equation,†

* Such an assumption is in good agreement with experiments made; see a paper by Dr. Seelmann, V.D.I. Vol. 69 (1925), p. 601, and F. Sass, Maschinenbau, Vol. 4, 1925, p. 1223. See also F. M. Lewis, loc. cit., p. 253.

† A detailed consideration of the twist of a crankshaft is given by the writer in Trans. Am. Soc. Mech. Eng., Vol. 44 (1922), p. 653. See also "Applied Elasticity," p. 188. Further discussion of this subject and also the bibliography can be found in the paper by R. Grammel, Ingenieur-Archiv, v. 4, p. 287, 1933, and in the doctor thesis by A. Kimmel, Stuttgart, 1935. There are also empirical formulae for the calculation of the equivalent length. See the paper by B. C. Carter, Engineering, v. 126, p. 36, 1928, and the paper by C. A. Norman and K. W. Stinson, S.A.E. journal, v. 23, p. 83, 1928.

$$l = C\left\{\frac{2b_1}{C_1} + \frac{a_1}{C_2}\left(1 - \frac{r}{k}\right) + \frac{2r}{B}\left(1 - \frac{r}{2k}\right)\right\}, \tag{87}$$

in which

$$k = \frac{\dfrac{r(a+h)^2}{4C_3} + \dfrac{ar^2}{2C_2} + \dfrac{a^3}{24B_1} + \dfrac{r^3}{3B} + \dfrac{1.2}{G}\left(\dfrac{a}{2F} + \dfrac{r}{F_1}\right)}{\dfrac{ar}{2C_2} + \dfrac{r^2}{2B}}, \tag{88}$$

$C_3 = \dfrac{c^3h^3G}{3.6(c^2 + h^2)}$ is the torsional rigidity of the web as a bar of rectangular cross section with sides h and c,

$B_1 = \dfrac{\pi d_2{}^4 E}{64}$ is the flexural rigidity of the crankpin,

F, F_1 are the cross sectional areas of the pin and of the web, respectively.

By taking $a_1 = 2b_1$ and $C_1 = C_2$ the complete constraint as it is seen from eqs. (86) and (87) reduces the equivalent length of shaft in the ratio $1 : \{1 - (r/2k)\}$. In actual conditions the length of the equivalent shaft will have an intermediate value between the two extreme cases considered above.

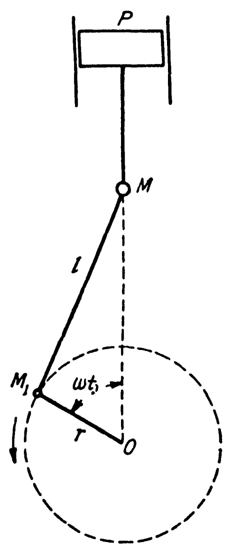

FIG. 154.

Another question to be decided in considering torsional vibration of crankshafts is the calculation of the inertia of the moving masses. Let us assume that the mass m of the connecting rod is replaced in the usual way * by two masses $m_1 = (I/l^2)$ at the crankpin and $m_2 = m - (I/l^2)$ at the cross head, where I denotes the moment of inertia of the connecting rod about the center of cross head. All other moving masses also can be replaced by masses concentrated in the same two points so that finally only two masses M and M_1 must be taken into consideration (Fig. 154). Let ω be constant angular velocity, ωt be the angle of the crank measured from the dead position as shown in Fig. 154. Then the velocity of the mass M_1 is equal to ωr and the velocity of the mass M, as shown in Art. 15 (see p. 78), is equal to

$$\omega r \sin \omega t + \frac{r^2\omega}{2l} \sin 2\omega t.$$

* See, for instance, "Regelung der Kraftmaschinen," by Max Tolle, 3d Ed. (1921) p. 116.

The kinetic energy of the moving masses of one crank will be

$$T = \tfrac{1}{2}M_1\omega^2r^2 + \tfrac{1}{2}M\omega^2r^2\left(\sin \omega t + \frac{r}{2l}\sin 2\omega t\right)^2.$$

The average value of T during one revolution is

$$T_0 = \frac{1}{2\pi}\int_0^{2\pi} Td(\omega t) = \frac{1}{2}\left\{M_1 + \frac{1}{2}M\left(1 + \frac{r^2}{4l^2}\right)\right\}\omega^2r^2.$$

By using this average value, the inertia of the moving parts connected with one crank can be replaced by the inertia of an equivalent disc having a moment of inertia

$$I = \left\{M_1 + \frac{1}{2}M\left(1 + \frac{r^2}{4l^2}\right)\right\}r^2.$$

By replacing all cranks by *equivalent lengths* of shaft and all moving masses by *equivalent discs* the problem on the vibration of crankshafts will be reduced to that of the torsional vibration of a cylindrical shaft and the critical speeds can be calculated as has been shown before.* It should be noted that such a method of investigating the vibration must be considered only as a rough approximation. The actual problem is much more complicated and in the simplest case of only one crank with a flywheel it reduces to a problem in torsional vibrations of a shaft with two discs, one of which has a variable moment of inertia. More detailed investigations show † that in such a system "forced vibrations" do not arise only from the pressure of the expanding gases on the piston. They are also produced by the incomplete balance of the reciprocating parts. Practically all the phenomena associated with dangerous critical speeds would appear if the fuel were cut off and the engine made to run without resistance at the requisite speed.

The positions of the critical speeds in such systems are approximately those found by the usual method, i.e., by replacing the moving masses by *equivalent discs.*‡

* Very often we obtain in this way a shaft with a comparatively large number of equal and equally spaced discs that replace the masses corresponding to individual cylinders, together with one or two larger discs replacing flywheels, generators, etc. For calculating critical speeds of such systems there exist numerical tables which simplify the work immensely. See R. Grammel, Ingenieur-Archiv, v. 2, p. 228, 1931 and v. 5, p. 83, 1934.

† See paper by G. R. Goldsbrough, "Torsional Vibration in Reciprocating Engine Shafts," Proc. of the Royal Society, Vol. 109 (1925), p. 99 and Vol. 113, 1927, p. 259.

‡ The bibliography on torsional vibration of discs of a variable moment of inertia is given on p. 160.

46. Damper with Solid Friction.—In order to reduce the amplitudes of torsional vibrations of crankshafts a damper with solid friction,* commonly known as the Lanchester damper, is very often used in gas and Diesel engines. The damper, Fig. 155, consists of two flywheels a free to rotate on bushings b, and driven by the crankshaft through friction rings c. The flywheels are pressed against these rings by means of loading springs and adjustable nuts d. If, due to resonance, large vibrations of the shaftend e and of the damper hub occur, the inertia of the flywheel prevents it from following the motion; the resultant relative motion between the hub and the flywheel gives rise to rubbing on the friction surfaces and a certain amount of energy will be dissipated.

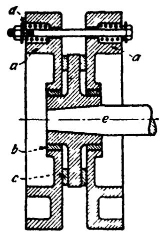

FIG. 155.

It was shown in the discussion of Art. 44 (see p. 268) that the amplitude of torsional vibration at resonance can be readily calculated if the amount of energy dissipated in the damper per cycle is known. To calculate this energy in the case of Lanchester damper, the motion of the damper flywheels must be considered. Under steady conditions the flywheels are rotating with an average angular velocity equal to the average angular velocity of the crankshaft. On this motion a motion relative to the oscillating

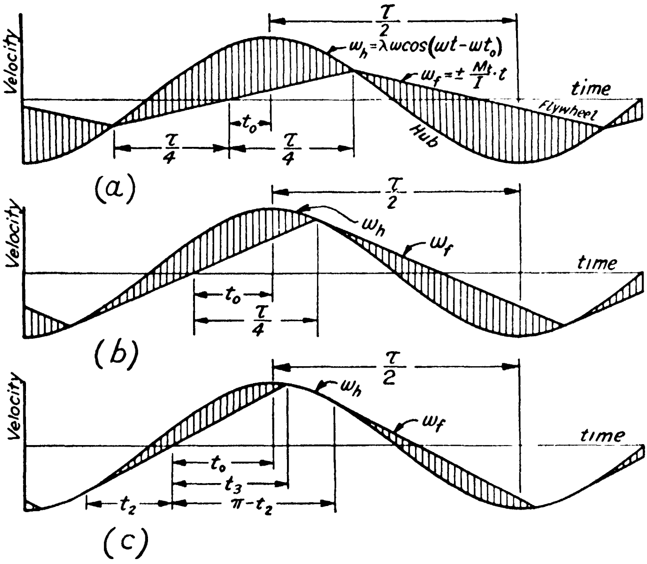

FIG. 156.

hub will be superimposed. It will be periodic motion and its frequency will be the same as that of the oscillating shaft. The three possible types of the superimposed motion are illustrated by the velocity diagrams in Fig. 156. The sinusoidal curves rep-

* The theory of this damper has been developed by J. P. Den Hartog and J. Ormondroyd, Trans. Amer. Soc. Mech. Engrs. v. 52, No. 22, p. 133, 1930.

resent the angular velocity ω_h of the oscillating hub. During slipping, the flywheel is acted upon by a constant friction torque M_f, therefore its angular velocity is a linear function of time, which is represented in the diagrams by straight lines. If the flywheel is slipping continuously we have the condition shown in Fig. 156*a*. The velocity ω_f* of the flywheel is represented by the broken line which shows that the flywheel has a periodically symmetrical motion. The velocity of this motion increases when the hub velocity ω_h is greater and decreases when the hub velocity is less than the flywheel velocity ω_f. The slopes of the straight lines are equal to the angular accelerations of the flywheel, i.e., equal to M_f/I where I is the total moment of inertia of the damper flywheels. As the damper loading springs are tightened up, the friction torque increases and the straight lines of the flywheel velocity diagram become steeper. Finally we arrive at the limiting condition shown in Fig. 156*b* when the straight line becomes tangent to the sine curve. This represents the limit of the friction torque below which slipping of the flywheel is continuous. If the friction is increased further, the flywheel clings to the hub until the acceleration of the hub is large enough to overcome the friction and we obtain the condition shown in Fig. 156*c*.

In our further discussion we assume that the damper flywheel is always sliding and we use the diagram in Fig. 156*a*. Noting that the relative angular velocity of the flywheel with respect to the hub is $\omega_f - \omega_h$, we see that the energy dissipated during an interval of time dt will be $M_f(\omega_h - \omega_f)dt$ so that the energy dissipated per cycle may be obtained by an integration:

$$E = \int_0^\tau M_f(\omega_h - \omega_f)dt, \tag{a}$$

where $\tau = 2\pi/\omega$ is the period of the torsional vibration of the shaft. In Fig. 156*a* this integral is represented to certain scale by the shaded area. In order to simplify the calculation of this area we take the time as being zero at the instant the superimposed velocity ω_f of the flywheel is zero and about to become positive, and we denote by t_0 the time corresponding to the maximum of the superimposed velocity ω_h of the hub. In this case the oscillatory motion of the hub is

$$\lambda \sin \omega(t - t_0),$$

and by differentiation we obtain

$$\omega_h = \lambda\, \omega \cos \omega(t - t_0). \tag{b}$$

The velocity of the flywheel for the interval of time $-\tau/4 < t < \tau/4$ will be

$$\omega_f = M_f t/I. \tag{c}$$

The time t_0 may be found from the condition that when $t = \pm\,\tau/4$ (see Fig. 156*a*) $\omega_f = \omega_h$. Then by using (*b*) and (*c*) we obtain

$$\frac{M_f}{I}\cdot\frac{\pi}{2\omega} = \lambda\omega \cos\left(\frac{\pi}{2} - \omega t_0\right) = \lambda\omega \sin \omega t_0,$$

and

$$\sin \omega t_0 = \frac{M_f}{I}\cdot\frac{\pi}{2\lambda\omega^2}. \tag{d}$$

* ω_f and ω_h denote the velocities of the flywheel and of the hub superimposed on the uniform average velocity of rotation of the crankshaft.

In calculating the amount of energy dissipated per cycle we note that the two shaded areas in Fig. 156*a* are equal. Hence

$$E = \int_0^{\tau} M_f(\omega_h - \omega_f)dt = 2\int_{-\tau/4}^{+\tau/4} M_f(\omega_h - \omega_f)dt,$$

or, substituting from (*b*) and (*c*),

$$E = 2M_f \int_{-\pi/2\omega}^{+\pi/2\omega} \left[\lambda\omega \cos \omega(t - t_0) - \frac{M_f t}{I}\right] dt.$$

Performing the integration we obtain

$$E = 4M_f\lambda \cos \omega t_0,$$

or by using (*d*) we find the final expression for the amount of energy dissipated per cycle:

$$E = 4M_f\lambda \sqrt{1 - \left(\frac{M_f}{I}\frac{\pi}{2\lambda\omega^2}\right)^2}. \tag{e}$$

By a change in the adjustable nuts *d* the friction torque M_f can be properly chosen. If the force exerted by the loading springs is very small the friction force is also small and its damping effect on the torsional vibrations of the crankshaft will be negligible. By tightening up the nuts we can get another extreme case when the friction torque is so large that the flywheel does not slide at all and no dissipation of energy takes place. The most effective damping action is obtained when the friction torque has the magnitude at which expression (*e*) becomes a maximum. Taking the derivative of this expression with respect to M_f and equating it to zero we find the most favorable value for the torque

$$M_f = \frac{\sqrt{2}}{\pi} \lambda\omega^2 I. \tag{f}$$

With this value substituted in (*e*) the energy dissipated per cycle becomes

$$E_{\max} = \frac{4}{\pi} \lambda^2\omega^2 I. \tag{g}$$

Having this expression we may calculate the amplitude of the forced vibration at resonance in the same manner as in the case of a viscous damping acting on one of the vibrating discs (see p. 268). If a pulsating torque $M \cos \omega(t - t_0)$ is acting on a disc of which the amplitude of torsional vibration is λ_m, the work done by this torque per cycle is (see p. 45) $M\lambda_m\pi$. Equating this work to the energy dissipated (*g*) we find

$$\lambda = \frac{\pi^2 M}{4\omega^2 I} \cdot \frac{\lambda_m}{\lambda}. \tag{h}$$

The ratio λ_m/λ can be taken from the normal elastic curve of the vibrating shaft so that if M and I are given the amplitude λ can be calculated from equation (*h*). Usually equation (*h*) may be applied for determining the necessary moment of inertia I of the damper. In such a case the amplitude λ should be taken of such a magnitude as to have the maximum torsional stress in the shaft below the allowable stress for the material of the shaft. Then the corresponding value of I may be calculated from equation (*h*).

47. Lateral Vibrations of Shafts on Many Supports.—*General.*—In our previous discussion (Art. 17) the simplest case of a shaft on two supports was considered and it was then shown that the critical speed of rotation of a shaft is that speed at which the number of revolutions per second is equal to the frequency of its natural lateral vibrations. In practice, however, cases of shafts on *many* supports are encountered and consideration will now be given to the various methods which may be employed for calculating the frequencies of the natural modes of lateral vibration of such shafts.*

Analytical Method.—This method can be applied without difficulty in the case of a shaft of uniform cross section carrying several discs.

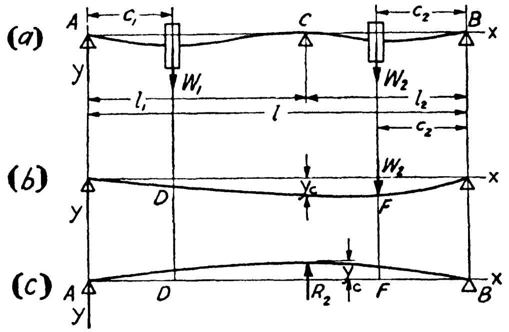

Fig. 157.

Let us consider first the simple example of a shaft on three supports carrying two discs (Fig. 157) the weights of which are W_1 and W_2. The statical deflections of the shaft under these loads can be represented by the equations,

$$\delta_1 = a_{11}W_1 + a_{12}W_2, \tag{a}$$

$$\delta_2 = a_{21}W_1 + a_{22}W_2, \tag{b}$$

the constants a_{11}, a_{12}, a_{21} and a_{22} of which can be calculated in the following manner. Remove the intermediate support C and consider the deflections produced by load W_2 alone (Fig. 157*b*); then the equation of the deflection curve for the left part of the shaft will be

$$y = \frac{W_2 c_2}{6lEI}(-x^3 + l^2 x - c_2{}^2 x), \tag{c}$$

* This subject is discussed in detail by A. Stodola, "Dampf- und Gasturbinen," 6th Ed., Berlin, 1924.

and the deflection at the point C becomes:

$$y_c = \frac{W_2c_2}{6lEI}(-l_1^3 + l^2l_1 - c_2^2l_1).$$

Now determine the reaction R_2 in such a manner as to reduce this deflection to zero (Fig. 157c). Applying eq. (c) for calculating the deflection under R_2 and putting this deflection equal to y_c, obtained above, we have,

$$\frac{W_2c_2}{6lEI}(-l_1^3 + l^2l_1 - c_2^2l_1) = \frac{R_2l_2}{6lEI}(-l_1^3 + l^2l_1 - l_2^2l_1),$$

from which

$$R_2 = \frac{W_2c_2(l^2 - l_1^2 - c_2^2)}{2l_1l_2^2}.$$

In the same manner the reaction R_1 produced by the load W_1 can be calculated and the complete reaction $R = R_1 + R_2$ at the middle support will be obtained. Now, by using eq. (c) the deflection y_1 produced by the loads W_1, W_2 and the reaction R can be represented in the form (a) in which

$$\begin{aligned} a_{11} &= \frac{1}{12ll_1^2EI}\left\{4l_1^2(l-c_1)^2c_1^2 - c_1(-c_1^3+l^2c_1-l_2^2c_1)(l^2-l_2^2-c_1^2)\right\}, \\ a_{12} &= \frac{1}{12ll_1l_2EI}\left\{2l_1l_2c_1c_2(l^2-c_1^2-c_2^2) - c_2c_1(l^2-l_2^2-c_1^2)(l^2-l_1^2-c_2^2)\right\}. \end{aligned} \qquad (d)$$

Interchanging l_2 and l_1 and c_2 and c_1 in the above equations, the constants a_{21} and a_{22} of eq. (b) will be obtained and it will be seen that $a_{12} = a_{21}$, i.e., that a load put at the location D produces at F the same deflection as a load of the same magnitude at F produces at D. Such a result should be expected on the basis of the *reciprocal theorem.*

Consider now the vibration of the loads W_1 and W_2 about their position of equilibrium, found above, and in the plane of the figure. Let y_1 and y_2 now denote the variable displacements of W_1 and W_2 from their positions of equilibrium during vibration. Then, neglecting the mass of the shaft, the kinetic energy of the system will be

$$T = \frac{W_1}{2g}(\dot{y}_1)^2 + \frac{W_2}{2g}(\dot{y}_2)^2. \qquad (e)$$

In calculating the increase in potential energy of the system due to displacement from the position of equilibrium equations (a) and (b) for static deflections will be used. Letting, for simplicity, $a_{11} = a$, $a_{12} = a_{21} = b$, $a_{22} = c$,* we obtain from the above equations (a) and (b) the following forces necessary to produce the deflections y_1 and y_2.

$$P_1 = \frac{cy_1 - by_2}{ac - b^2}; \qquad P_2 = \frac{ay_2 - by_1}{ac - b^2};$$

and

$$V = \frac{P_1 y_1}{2} + \frac{P_2 y_2}{2} = \frac{1}{2(ac - b^2)}(cy_1^2 - 2by_1y_2 + ay_2^2). \tag{f}$$

Substituting (e) and (f) in Lagrange's equations (73) we obtain the following differential equations for the free lateral vibration of the shaft

$$\begin{aligned} &\frac{W_1}{g}\ddot{y}_1 + \frac{c}{ac - b^2}y_1 - \frac{b}{ac - b^2}y_2 = 0, \\ &\frac{W_2}{g}\ddot{y}_2 - \frac{b}{ac - b^2}y_1 + \frac{a}{ac - b^2}y_2 = 0. \end{aligned} \tag{g}$$

Assuming that the shaft performs one of the natural modes of vibration and substituting in eqs. (g):

$$y_1 = \lambda_1 \cos pt; \qquad y_2 = \lambda_2 \cos pt,$$

we obtain

$$\begin{aligned} &\lambda_1\left(\frac{c}{ac - b^2} - \frac{W_1}{g}p^2\right) - \frac{b}{ac - b^2}\lambda_2 = 0, \\ &-\frac{b}{ac - b^2}\lambda_1 + \lambda_2\left(\frac{a}{ac - b^2} - \frac{W_2}{g}p^2\right) = 0. \end{aligned} \tag{h}$$

By putting the determinant of these equations equal to zero the following *frequency equation* will be obtained

$$\left(\frac{c}{ac - b^2} - \frac{W_1}{g}p^2\right)\left(\frac{a}{ac - b^2} - \frac{W_2}{g}p^2\right) - \frac{b^2}{(ac - b^2)^2} = 0, \tag{k}$$

from which

$$p^2 = \frac{g}{2(ac - b^2)}\left\{\frac{c}{W_1} + \frac{a}{W_2} \pm \sqrt{\left(\frac{c}{W_1} + \frac{a}{W_2}\right)^2 - \frac{4(ac - b^2)}{W_1W_2}}\right\}. \tag{89}$$

In this manner two positive roots for p^2, corresponding to the two principal modes of vibration of the shaft are obtained. Substituting these two roots

* The constants a, b and c can be calculated for any particular case by using eqs. (d).

in one of the eqs. (h) two different values for the ratio λ_1/λ_2 will be obtained. For the larger value of p^2 the ratio λ_1/λ_2 becomes positive, i.e., both discs during the vibration move simultaneously in the same direction and the mode of vibration is as shown in Fig. 158a. If the smaller root of p^2 be substituted in eq. (h) the ratio λ_1/λ_2 becomes negative and the corresponding mode of vibration will be as shown in Fig. 158b. Take, for instance, the particular case when (see Fig. 157) $W_1 = W_2$; $l_1 = l_2 = (l/2)$ and

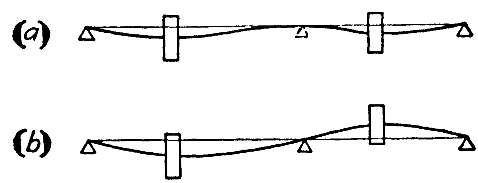

FIG. 158.

$c_1 = c_2 = (l/4)$. Substituting in eqs. (d) and using the conditions of symmetry, we obtain:

$$a = c = \frac{23}{48 \times 256}\frac{l^3}{EI} \quad \text{and} \quad b = -\frac{9}{48 \times 256}\frac{l^3}{EI}.$$

Substituting in eq. (89), we have

$$p_1{}^2 = \frac{g}{(a-b)W} = \frac{g48EI}{W(l/2)^3}; \qquad p_2{}^2 = \frac{g768EI}{7W(l/2)^3}.$$

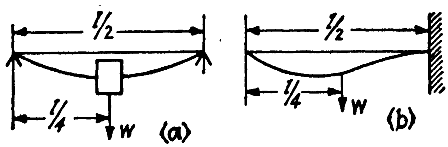

FIG. 159.

These two frequencies can also be easily derived by substituting in eq. 5 (see p. 3) the statical deflections

$$\delta'_{st} = \frac{W(l/2)^3}{48EI} \quad \text{and} \quad \delta''_{st} = \frac{7W(l/2)^3}{768EI}$$

for the cases shown in Fig. 159.

Another method of solution of the problem on the lateral vibrations of shafts consists in the application of D'Alembert's principle. In using this principle the equations of vibration will be written in the same manner as the equations of statics. It is only necessary to add to the loads acting on the shaft the inertia forces. Denoting as before, by y_1 and y_2 the deflections of the shaft from the position of equilibrium under the loads W_1 and W_2, respectively, the inertia forces will be $-(W_1/g)\ddot{y}_1$ and $-(W_2/g)\ddot{y}_2$. These inertia forces must be in equilibrium with the elastic forces due to the additional deflection and two equations equivalent to (a) and (b) can be written down as follows.

$$y_1 = -a\frac{W_1}{g}\ddot{y}_1 - b\frac{W_2}{g}\ddot{y}_2,$$

$$y_2 = -b\frac{W_1}{g}\ddot{y}_1 - c\frac{W_2}{g}\ddot{y}_2. \tag{l}$$

Assuming, as before,

$$y_1 = \lambda_1 \cos pt; \qquad y_2 = \lambda_2 \cos pt,$$

and substituting in eqs. (l) we obtain,

$$\lambda_1\left(1 - \frac{aW_1}{g}p^2\right) - \lambda_2 b\frac{W_2}{g}p^2 = 0,$$

$$-\lambda_1 b\frac{W_1}{g}p^2 + \lambda_2\left(1 - \frac{cW_2}{g}p^2\right) = 0. \tag{m}$$

Putting the determinant of these two equations equal to zero, the frequency equation (k), which we had before, will be obtained.

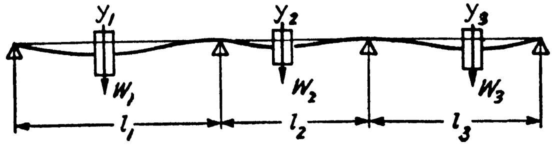

FIG. 160.

The methods developed above for calculating the frequencies of the lateral vibrations can be used also in cases where the number of discs or the number of spans is greater than two. Take, for instance, the case shown in Fig. 160. By using a method analogous to the one employed in the previous example, the statical deflections of the shaft under the discs can be represented in the following form:

$$\begin{aligned}\delta_1 &= a_{11}W_1 + a_{12}W_2 + a_{13}W_3,\\ \delta_2 &= a_{21}W_1 + a_{22}W_2 + a_{23}W_3,\\ \delta_3 &= a_{31}W_1 + a_{32}W_2 + a_{33}W_3,\end{aligned} \tag{n}$$

in which a_{11}, a_{12}, . . . are constants depending on the distances between the supports, the distances of the discs from the supports and on the flexural rigidity of the shaft. From the *reciprocal theorem* it can be concluded at once that $a_{12} = a_{21}$, $a_{13} = a_{31}$ and $a_{23} = a_{32}$. Applying now D'Alembert's principle and denoting by y_1, y_2 and y_3 the displacements of the discs during vibration from the position of equilibrium, the following equations of vibration will be derived from the statical equations (n).

$$
\begin{aligned}
y_1 &= - a_{11}\frac{W_1}{g}\ddot{y}_1 - a_{12}\frac{W_2}{g}\ddot{y}_2 - a_{13}\frac{W_3}{g}\ddot{y}_3, \\
y_2 &= - a_{21}\frac{W_1}{g}\ddot{y}_1 - a_{22}\frac{W_2}{g}\ddot{y}_2 - a_{23}\frac{W_3}{g}\ddot{y}_3, \\
y_3 &= - a_{31}\frac{W_1}{g}\ddot{y}_1 - a_{32}\frac{W_2}{g}\ddot{y}_2 - a_{33}\frac{W_3}{g}\ddot{y}_3,
\end{aligned}
$$

from which the *frequency equation*, a cubic in p^2, can be gotten in the usual manner. The three roots of this equation will give the frequencies of the three principal modes of vibration of the system under consideration.*

It should be noted that the frequency equations for the lateral vibrations of shafts can be used also for calculating *critical speeds* of rotation. A critical speed of rotation is a speed at which the centrifugal forces of the rotating masses are sufficiently large to keep the shaft in a bent condition (see Art. 17). Take again the case of two discs (Fig. 155a) and assume that y_1 and y_2 are the deflections, produced by the centrifugal forces † $(W_1/g)\omega^2 y_1$ and $(W_2/g)\omega^2 y_2$ of the rotating discs. Such deflections can exist only if the centrifugal forces satisfy the following conditions of equilibrium [see eqs. (a) and (b)],

$$
\begin{aligned}
y_1 &= a_{11}\frac{W_1}{g}\omega^2 y_1 + a_{12}\frac{W_2}{g}\omega^2 y_2, \\
y_2 &= a_{21}\frac{W_1}{g}\omega^2 y_1 + a_{22}\frac{W_2}{g}\omega^2 y_2.
\end{aligned}
\tag{o}
$$

These equations can give for y_1 and y_2 solutions different from zero only in the case when their determinant vanishes. Observing that the equa-

* A graphical method of solution of frequency equations has been developed by C. R. Soderberg, Phil. Mag., Vol. 5, 1928, p. 47.

† The effect of the weight of the shaft on the critical speeds will be considered later.

tions (o) are identical with the equations (m) above and equating their determinant to zero, an equation identical with eq. (k) will be obtained for calculating the critical speeds of rotation.

Graphical Method.—In the case of shafts of variable cross section or those having many discs the analytical method of calculating the critical speeds, described above, becomes very complicated and recourse should be made to graphical methods. As a simple example, a shaft supported at the ends will now be considered (Fig. 62). Assume some *initial deflection* of the rotating shaft satisfying the end conditions where y_1, y_2, . . . are the deflections at the discs W_1, W_2, . . . If ω be the angular velocity then the corresponding centrifugal forces will be $(W_1/g)\omega^2 y_1$, $(W_2/g)\omega^2 y_2$, . . . Considering these forces as statically applied to the shaft, the corresponding deflection curve can be obtained graphically as was explained in Art. 17. If our assumption about the shape of the *initial* deflection curve was correct, the deflections y_1', y_2', . . ., as obtained *graphically*, should be proportional to the deflections y_1, y_2, . . . initially assumed, and the critical speed will be found from the equation

$$\omega_{cr} = \omega \sqrt{\frac{y_1}{y_1'}}. \tag{90}$$

This can be explained in the following manner.

By taking ω_{cr} as given by (90) instead of ω, in calculating the centrifugal forces as above, all these forces will increase in the ratio y_1/y_1'; the deflections graphically derived will now also increase in the same proportion and the deflection curve, as obtained graphically, will now coincide with the *initially assumed* deflection curve. This means that at a speed given by eq. (90), the centrifugal forces are sufficient to keep the rotating shaft in a deflected form. Such a speed is called a *critical speed* (see p. 282).

It was assumed in the previous discussion that the deflection curve, as obtained graphically, had deflections proportional to those of the curve initially taken. If there is a considerable difference in the shape of these two curves and a closer approximation for ω_{cr} is desired, the construction described above should be repeated by taking the deflection curve, obtained graphically, as the initial deflection curve.*

The case of a shaft on three supports and having one disc on each span (Fig. 157) will now be considered. In the solution of this problem we may

* It was pointed out in considering Rayleigh's method (see Art. 16), that a considerable error in the shape of the assumed deflection curve produces only a small effect on the magnitude of ω_{cr} provided the conditions at the ends are satisfied.

proceed exactly in the same manner as before in the analytical solution and establish first the equations,

$$\delta_1 = a_{11}W_1 + a_{12}W_2, \qquad (a)'$$

$$\delta_2 = a_{21}W_1 + a_{22}W_2, \qquad (b)'$$

between the acting forces and the resultant deflections.

In order to obtain the values of the constants a_{11}, a_{12}, . . . graphically we assume first that the load W_1 is acting alone and that the middle support is removed (Fig. 161a); then the deflections y_1', y_2' and y_c' can easily be obtained by using the graphical method, described before (see p. 95). Now, by using the same method, the deflection curve produced by a vertical force R' applied at C and acting in an upward direction should be constructed and the deflections y_1'', y_c'' and y_2'' measured. Taking into consideration that the deflection at the support C should be equal to zero the reaction R of this support will now be found from the equation,

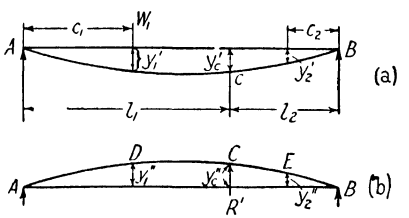

FIG. 161.

$$R = R'\frac{y_c'}{y_c''}\ * \qquad (p)$$

and the actual deflections at D and E, produced by load W_1, will be

$$y_{11} = y_1' - y_1''\frac{y_c'}{y_c''}, \qquad (q)$$

$$y_{21} = y_2' - y_2''\frac{y_c'}{y_c''}.$$

Comparing these equations with the eqs. $(a)'$ and $(b)'$ we obtain

$$a_{11}W_1 = y_1' - y_1''\frac{y_c'}{y_c''},$$

$$a_{21}W_1 = y_2' - y_2''\frac{y_c'}{y_c''},$$

* Absolute values of the deflections are taken in this equation.

from which the constants a_{11} and a_{21} can be calculated. In the same manner, considering the load W_2, the constants a_{12} and a_{22} can be found. All constants of eqs. $(a)'$ and $(b)'$ being determined, the two critical speeds of the shaft can be calculated by using formula (89), in which $a = a_{11}$; $b = a_{12} = a_{21}$; $c = a_{22}$.

In the previous calculations, the reaction R at the middle support has been taken as the statically indeterminate quantity. In case there are many supports, it is simpler to take as statically indeterminate quantities the bending moments at the intermediate supports. To illustrate this method of calculation, let us consider a motor generator set consisting of an induction motor and a D.C. generator supported on three bearings.* The dimensions of the shaft of variable cross section are given in figure 162 (a). We assume that the masses of the induction motor armature, D.C. armature and also D.C. commutator are concentrated at their centers of gravity (Fig. 162a). In order to take into account the mass of the shaft, one-half of the mass of the left span of the shaft has been added to the mass of the induction motor and one-half of the mass of the right span of the shaft has been equally distributed between the D.C. armature and D.C. commutator. In this manner the problem is reduced to one of three degrees of freedom and the deflections y_1, y_2, y_3 of the masses W_1, W_2, and W_3 during vibration will be taken as coordinates. The statical deflections under the action of loads W_1, W_2, W_3 can be represented by eqs. (n) and the constants a_{11}, a_{12}, . . . of these eqs. will now be determined by taking the bending moment at the intermediate support as the statically indeterminate quantity. In order to obtain a_{11}, let us assume that the shaft is cut into two parts at the intermediate support and that the right span is loaded by a 1 lb. load at the cross section where W_1 is applied (Fig. 162b). By using the graphical method, explained in Art. 17, we obtain the deflection under the load $a_{11}' = 2.45 \times 10^{-6}$ inch and the slope at the left support $\gamma_1 = 5.95 \times 10^{-8}$ radian. By applying now a bending moment of 1 inch pound at the intermediate support and using the same graphical method, we obtain the slopes $\gamma_2 = 4.23 \times 10^{-9}$ (Fig. 162c) and $\gamma_3 = 3.5 \times 10^{-9}$ (Fig. 162d). From the *reciprocity theorem* it follows that the deflection at the point W_1 for this case is numerically equal to the slope γ_1, in the case shown in Fig. 162b. Combining these results it can now be concluded that the bending moment at the intermediate support produced by a

* These numerical data represent an actual case calculated by J. P. DenHartog, Research Engineer, Westinghouse Electric and Manufacturing Company, East Pittsburgh, Pennsylvania.

load of 1 lb. at the point W_1 is

$$M = \frac{\gamma_1}{\gamma_2 + \gamma_3} \text{ lbs.} \times \text{inch},$$

and that the deflection under this load is

$$a_{11} = a_{11}' - \frac{\gamma_1^2}{\gamma_2 + \gamma_3} = 19.9 \times 10^{-7}.$$

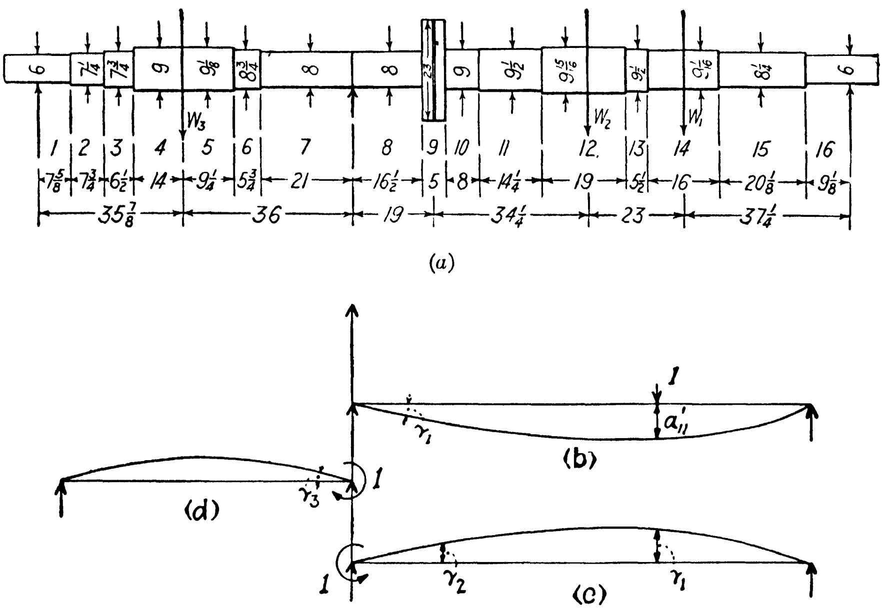

FIG. 162.

Proceeding in the same manner with the other constants of eqs. (n) the following numerical values have been obtained:

$$a_{22} = 19.6 \times 10^{-7}; \qquad a_{33} = 7.6 \times 10^{-7}; \qquad a_{12} = a_{21} = 18.1 \times 10^{-7};$$

$$a_{13} = a_{31} = -3.5 \times 10^{-7}; \qquad a_{23} = a_{32} = -4.6 \times 10^{-7}.$$

Now substituting in eqs. (n) the centrifugal forces $W_1\omega^2 y_1/g$, $W_2\omega^2 y_2/g$ and $W_3\omega^2 y_3/g$ instead of the loads W_1, W_2, W_3, the following equations will be found.

$$\left(1 - a_{11}\frac{W_1\omega^2}{g}\right)y_1 - a_{12}\frac{W_2\omega^2}{g}y_2 - a_{13}\frac{W_3\omega^2}{g}y_3 = 0,$$

$$-a_{21}\frac{W_1\omega^2}{g}y_1+\left(1-a_{22}\frac{W_2\omega^2}{g}\right)y_2-a_{23}\frac{W_3\omega^2}{g}y_3=0,$$

$$-a_{31}\frac{W_1\omega^2}{g}y_1-a_{32}\frac{W_2\omega^2}{g}y_2+\left(1-a_{33}\frac{W_3\omega^2}{g}\right)y_3=0.$$

If the determinant of this system of equations be equated to zero, and the quantities calculated above be used for the constants a_{11}, a_{12}, $\cdots$ the following frequency equation for calculating the critical speeds is arrived at:

$$(\omega^2 10^{-7})^3-3.76(\omega^2 10^{-7})^2+1.93(\omega^2 10^{-7})-.175=0,$$

from which the three critical speeds in R.P.M. are:

$$n_1=\frac{\omega_1 60}{2\pi}=1070;\qquad n_2=\frac{\omega_2 60}{2\pi}=2240;\qquad n_3=\frac{\omega_3 60}{2\pi}=5620.$$

In addition to the above method, the direct method of graphical solution previously described for a shaft with one span, can also be applied to the present case of two spans. In this case an initial deflection curve satisfying the conditions at the supports (Fig. 158, a, b) should be taken and a certain angular velocity ω assumed. The centrifugal forces acting on the shaft will then be

$$\frac{W_1}{g}\omega^2 y_1 \qquad \text{and} \qquad \frac{W_2}{g}\omega^2 y_2.$$

By using the graphical method the deflection curve produced by these two forces can be constructed and if the initial curve was chosen correctly the constructed deflection curve will be geometrically similar to the initial curve and the critical speed will be obtained from an equation analogous to eq. (90). If there is a considerable difference in the shape of these two curves the construction should be repeated by considering the obtained deflection curve as the initial curve.*

This method can be applied also to the case of many discs and to cases where the mass of the shaft should be taken into consideration. We begin again by taking an initial deflection curve (Fig. 163) and by assuming a certain angular velocity ω. Then the centrifugal forces P_1, P_2, $\cdots$ acting on the discs and on portions of the shaft can easily be calculated, and the

* It can be shown that this process is convergent when calculating the slowest critical speed and by repeating the construction described above we approach the true critical speed. See the book by A. Stodola, "Dampf- und Gasturbinen," 6th Ed. 1924. Berlin.

corresponding deflection curve can be constructed as follows: Consider first the forces acting on the left span of the shaft and, removing the middle support C, construct the deflection curve shown in Fig. 163*b*. In the same manner the deflection curve produced by the vertical load R' applied at C and acting in an upward direction can be obtained (Fig. 163*c*) and reaction R at the middle support produced by the loading of the left span of the shaft can be calculated by using eq. (p) above. The deflection produced at any point by the loading of the left side of the shaft can then be found by using equations, similar to equations (q).

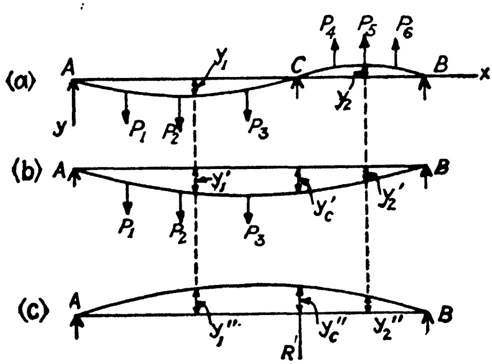

FIG. 163.

Taking, for instance, the cross sections in which the initial curve has the largest deflections y_1 and y_2 (Fig. 163*a*) the deflections produced at these cross sections by the loading acting on the left side of the shaft will be

$$y_{1a} = y_1' - y_1''\frac{y_c'}{y_c''},$$

$$y_{2a} = y_2' - y_2''\frac{y_c'}{y_c''}.$$

In the same manner the deflections y_{1b} and y_{2b} produced in these cross sections by the loading of the right side of the shaft can be obtained and the complete deflections $y_{1a} + y_{1b}$ and $y_{2a} + y_{2b}$ can be calculated.* If the initial deflection curve was chosen correctly, the following equation should be fulfilled:

$$\frac{y_{1a} + y_{1b}}{y_{2a} + y_{2b}} = \frac{y_1}{y_2}, \tag{r}$$

* Deflections in a downward direction are taken as positive.

and the critical speed will be calculated from the equation

$$\omega_{cr} = \omega \sqrt{\frac{y_1}{y_{1a} + y_{1b}}}. \tag{91}$$

If there is a considerable deviation from condition (r) the calculation of a second approximation becomes necessary for which purpose the following procedure can be adopted.* It is easy to see that the deflections y_{1a} and y_{2a}, found above, should be proportional to ω^2 and to the initial deflection y_1, so that we can write the equations

$$y_{1a} = a_1 y_1 \omega^2,$$
$$y_{2a} = a_2 y_1 \omega^2,$$

from which the constants a_1 and a_2 can be calculated. In the same manner from the equations

$$y_{1b} = b_1 y_2 \omega^2,$$
$$y_{2b} = b_2 y_2 \omega^2,$$

the constants b_1 and b_2 can be found.

Now, if the initial deflection curve had been chosen correctly and if $\omega = \omega_{cr}$, the following equations should be satisfied

$$y_1 = y_{1a} + y_{1b} = a_1 y_1 \omega^2 + b_1 y_2 \omega^2,$$
$$y_2 = y_{2a} + y_{2b} = a_2 y_1 \omega^2 + b_2 y_2 \omega^2,$$

which can be written as follows:

$$\begin{aligned} (1 - a_1\omega^2)y_1 - b_1\omega^2 y_2 &= 0, \\ - a_2 y_1 \omega^2 + (1 - b_2\omega^2)y_2 &= 0. \end{aligned} \tag{s}$$

The equation for calculating the critical speed will now be obtained by equating to zero the determinant of these equations, and we obtain,

$$(a_1 b_2 - a_2 b_1)\omega^4 - (a_1 + b_2)\omega^2 + 1 = 0.$$

The root of this equation which makes the ratio y_1/y_2 of eqs. (s) negative, corresponds to the assumed shape of the curve (Fig. 163a) and gives the lowest critical speed. For obtaining a closer approximation the ratio y_1/y_2, as obtained from eqs. (s), should be used in tracing the new shape of the

* This method was developed by Mr. Borowicz in his thesis "Beiträge zur Berechnung krit. Geschwindigkeiten zwei und mehrfach gelagerter Wellen," München, 1915. See also E. Rausch, Ingenieur-Archiv, Vol. I, 1930, p. 203., and the book by K. Karas, "Die Kritische Drehzahlen Wichtiger Rotorformen," 1935, Berlin.

initial curve and with this new curve the graphical solution should be repeated. In actual cases this further approximation is usually unnecessary.

48. Gyroscopic Effects on the Critical Speeds of Rotating Shafts.—*General.*—In our previous discussion on the critical speeds of rotating shafts only the centrifugal forces of the rotating masses were taken into consideration. Under certain conditions not only these forces, but also the moments of the inertia forces due to angular movements of the axes of the rotating masses are of importance and should be taken into account in calculating the critical speeds. In the following the simplest case of a single circular disc on a shaft will be considered (Fig. 164).

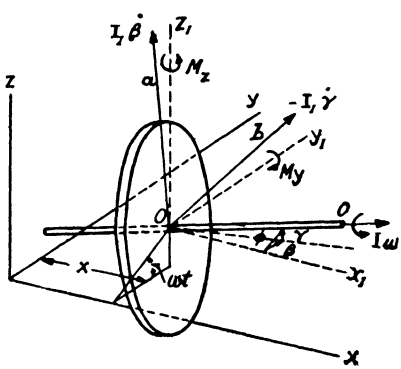

FIG. 164

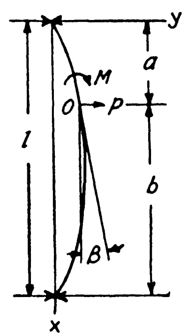

FIG. 165.

Assuming that the deflections y and z of the shaft during vibration are very small and that the center of gravity O of the disc coincides with the axis of the shaft, the position of the disc will be completely determined by the coordinates y and z of its center and by the angles β and γ which the axis O–O perpendicular to the plane of the disc and tangent to the deflection curve of the shaft makes with the fixed planes xz and xy, perpendicular to each other and drawn through the x axis joining the centers of the bearings. Letting W equal the weight of the disc and taking into consideration the elastic reactions of the shaft * only, the equations of motion of the center of gravity of the disc will be

$$\frac{W}{g}\ddot{y} = Y, \qquad \frac{W}{g}\ddot{z} = Z, \tag{a}$$

* The conditions assumed here correspond to the case of a vertical shaft when the weight of the disc does not affect the deflections of the shaft. The effect of this weight will be considered later (see p. 299).

in which Y and Z are the components of the reaction of the shaft in the y and z directions. These reactions are linear functions of the coordinates y, z and of the angles β, γ which can be determined from the consideration of the bending shaft.

Take, for instance, the bending of a shaft with simply supported ends, in the xy plane (Fig. 165) under the action of a force P and of a couple M. Considering in the usual way the deflection curve of the shaft we obtain the deflection at O equal to *

$$y = \frac{Pa^2b^2}{3lB} + \frac{Mab(a-b)}{3lB}, \tag{b}$$

and the slope at the same point equal to

$$\beta = \frac{Pab(b-a)}{3lB} - \frac{M(a^2 - ab + b^2)}{3lB} \tag{c}$$

where B is the flexural rigidity of the shaft.
From eqs. (b) and (c) we obtain

$$P = 3lB\left(\frac{a^2 - ab + b^2}{a^3b^3}y + \frac{a-b}{a^2b^2}\beta\right), \tag{d}$$

$$M = 3lB\left(\frac{b-a}{a^2b^2}y - \frac{1}{ab}\beta\right). \tag{e}$$

By using eq. (d) the eqs. (a) of motion of the center of gravity of the disc become

$$\frac{W}{g}\ddot{y} + my + n\beta = 0; \qquad \frac{W}{g}\ddot{z} + mz + n\gamma = 0, \tag{92}$$

in which

$$m = 3lB\frac{a^2 - ab + b^2}{a^3b^3}; \qquad n = 3lB\frac{a-b}{a^2b^2}. \tag{h}$$

In considering the relative motion of the disc about its center of gravity it will be assumed that the moment of the external forces acting on the disc with respect to the O–O axis is always equal to zero, then the angular velocity ω with respect to this axis remains constant. The moments M_y and M_z, taken about the y_1 and z_1 axes parallel to the y and z axes (see

* See "Applied Elasticity" p. 89.

Fig. 164), and representing the action of the elastic forces of the shaft on the disc can be written in the following form,

$$\begin{aligned} M_y &= -m'z + n'\gamma, \\ M_z &= m'y - n'\beta, \end{aligned} \tag{g}$$

in which m' and n' are constants which can be obtained from the deflection curve of the shaft.* The positive directions for the angles β and γ and for the moments M_y and M_z are indicated in the figure.

In the case considered above (see eq. *e*), we have

$$m' = 3lB\frac{b-a}{a^2b^2}; \qquad n' = \frac{3lB}{ab}. \tag{k}$$

The equations of relative motion of the disc with respect to its center of gravity will now be obtained by using the principle of *angular momentum* which states that the *rate of increase* of the total moment of momentum of any moving system about any *fixed* axis is equal to the total moment of the external forces about this axis. In calculating the *rate of change* of the angular momentum about a fixed axis drawn through the instantaneous position of center of gravity O we may take into consideration only the relative motion.†

In calculating the components of the angular momentum the principal axis of inertia of the disc will be taken. The axis of rotation OO is one of these axes. The two other axes will be two perpendicular diameters of the disc. One of these diameters Oa we taken in the plane OOz_1 (see Fig. 164). It will make a small angle γ with the axis Oz_1. Another diameter Ob will make the angle β with the axis Oy_1.

Let I = moment of inertia of the disc about the O–O axis,

$I_1 = I/2$ = moment of inertia of the disc about a diameter.

Then the component of angular momentum about the OO axis will be $I\omega$, and the components about the diameters Oa and Ob will be $I_1\dot{\beta}$ and $-I_1\dot{\gamma}$, respectively.‡ Positive directions of these components of the angular momentum are shown in Fig. 164. Projecting these components on the fixed axes Oy_1 and Oz_1 through the instantaneous position of the center of

* It is assumed that the flexibility of the shaft including the flexibility of its supports is the same in both directions.

† See, for instance, H. Lamb, "Higher Mechanics," 1920, p. 94.

‡ It is assumed, as before, that β and γ are small. Then $\dot{\beta}$ and $-\dot{\gamma}$ will be approximate values of the angular velocities about the diameters Oa and Ob.

gravity O we obtain $I\omega\beta - I_1\dot{\gamma}$ and $I\omega\gamma + I_1\dot{\beta}$, respectively. Then from the principle of angular momentum we have

$$\frac{d}{dt}(I\omega\beta - I_1\dot{\gamma}) = M_y \quad \text{and} \quad \frac{d}{dt}(I\omega\gamma + I_1\dot{\beta}) = M_z,$$

or, by using eqs. (g),

$$\begin{aligned} I\omega\dot{\beta} - I_1\ddot{\gamma} &= -m'z + n'\gamma, \\ I\omega\dot{\gamma} + I_1\ddot{\beta} &= m'y - n'\beta. \end{aligned} \tag{93}$$

Four eqs. (92) and (93) describing the motion of the disc, will be satisfied by substituting

$$y = A\sin pt; \quad z = B\cos pt; \quad \beta = C\sin pt; \quad \gamma = D\cos pt. \tag{m}$$

In this manner four linear homogeneous equations in A, B, C, D will be obtained. Putting the determinant of these equations equal to zero, the equation for calculating the frequencies p of the natural vibrations will be determined.* Several particular cases will now be considered.

As a first example consider the case in which the principal axis OO perpendicular to the plane of the disc remains always in a plane containing the x axis and rotating with the constant angular velocity ω, with which the disc rotates. Putting r equal to the deflection of the shaft and φ equal to the angle between OO and x axes (see Fig. 164) we obtain for this particular case,

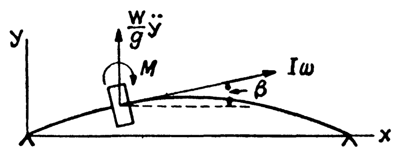

FIG. 166.

$$y = r\cos\omega t; \quad z = r\sin\omega t; \quad \beta = \varphi\cos\omega t; \quad \gamma = \varphi\sin\omega t. \tag{n}$$

Considering r and φ as constants and considering the instantaneous position when the plane of the deflected line of the shaft coincides with the xy plane (Fig. 166) we obtain from eqs. (n)

$$\begin{array}{lll} \beta = \varphi, & \dot{\beta} = 0, & \ddot{\beta} = -\varphi\omega^2, \\ \gamma = 0, & \dot{\gamma} = \varphi\omega, & \ddot{\gamma} = 0, \\ y = r, & \dot{y} = 0, & \ddot{y} = -r\omega^2, \\ z = 0, & \dot{z} = r\omega, & \ddot{z} = 0. \end{array}$$

* See the paper by A. Stodola in Zeitschrift, f. d. gesamte Turbinenwesen, 1918, p. 253, and 1920, p. 1.

Substituting in eqs. (92) and (93) we obtain,

$$\frac{W}{g}\ddot{y} + my + n\beta = 0,$$

$$(I - I_1)\beta\omega^2 = m'y - n'\beta. \tag{o}$$

It is seen that the shaft is bent not only by centrifugal force but also by the moment $M = (I - I_1)\beta\omega^2$ which represents the *gyroscopic effect* of the rotating disc in this case and makes the shaft stiffer. Substituting

$$y = r\cos\omega t, \qquad \beta = \varphi\cos\omega t,$$

in eqs. (o) we obtain,

$$\left(m - \omega^2\frac{W}{g}\right)r + n\varphi = 0,$$

$$-m'r + \left\{n' + (I - I_1)\omega^2\right\}\varphi = 0. \tag{p}$$

The deflection of the shaft, assumed above, becomes possible if eqs. (p) may have for r and φ roots other than zero, i.e., when the angular velocity ω is such that the determinant of these equations becomes equal to zero. In this manner the following equation for calculating the critical speeds will be found:

$$\left(m - \frac{\omega^2 W}{g}\right)\left\{n' + (I - I_1)\omega^2\right\} + nm' = 0, \tag{r}$$

or letting

$$\frac{mg}{W} = p^2; \qquad \frac{n'}{I - I_1} = q^2,$$

and noting that, from (h) and (k),

$$nm' = -cmn', \qquad \text{where } c = \frac{(a-b)^2}{a^2 - ab + b^2},$$

we obtain

$$(p^2 - \omega^2)(q^2 + \omega^2) - cp^2q^2 = 0$$

or

$$\omega^4 - (p^2 - q^2)\omega^2 - p^2q^2(1 - c) = 0. \tag{s}$$

It is easy to see that (for $c < 1$) eq. (s) has only one positive root for ω^2, namely,

$$\omega^2 = \tfrac{1}{2}(p^2 - q^2) + \sqrt{\tfrac{1}{4}(p^2 - q^2)^2 + (1 - c)p^2q^2}. \tag{t}$$

When the gyroscopic effect can be neglected, $I - I_1 = 0$ should be substituted in (r) and we obtain,

$$\frac{\omega^2 W}{g} = \frac{mn' + nm'}{n'} = \frac{3lB}{a^2b^2},$$

from which

$$\omega_{cr} = \sqrt{\frac{g3lB}{a^2b^2W}} \quad \text{or} \quad \omega_{cr} = \sqrt{\frac{g}{\delta}},$$

where

$$\delta = \frac{a^2b^2W}{3lB}$$

represents the statical deflection of the shaft under the load W. This result coincides completely with that found before (see Art. 17) considering the disc on the shaft as a system with one degree of freedom.

In the above discussion it was assumed that the angular velocity of the plane of the deflected shaft is the same as that of the rotating disc. It is possible also that these two velocities are different. Assuming, for instance, that the angular velocity of the plane of the deflected shaft is ω_1 and substituting,

$$y = r\cos\omega_1 t; \qquad z = r\sin\omega_1 t; \qquad \beta = \varphi\cos\omega_1 t; \qquad \gamma = \varphi\sin\omega_1 t$$

in eqs. (f) and (l) we obtain,

$$\begin{aligned} \frac{W}{g}\ddot{y} + my + n\beta &= 0, \\ (I\omega\omega_1 - I_1\omega_1^2)\beta &= m'y - n'\beta, \end{aligned} \qquad (o)^1$$

instead of eqs. (o).

By putting $\omega_1 = \omega$ the previous result will be obtained. If $\omega_1 = -\omega$ we obtain from the second of eqs. $(o)^1$

$$-(I + I_1)\omega^2\beta = m'y - n'\beta. \qquad (u)$$

This shows that when the plane of the bent shaft rotates with the velocity ω in the direction opposite to that of the rotation of the disc, the gyroscopic effect will be represented by the moment

$$M = -(I + I_1)\omega^2\beta.$$

The minus sign indicates that under such conditions the gyroscopic moment is acting in the direction of increasing the deflection of the shaft and hence lowers the critical speed of the shaft. If the shaft with the disc is brought up to the speed ω from the condition of rest, the condition $\omega_1 = \omega$ usually takes place. But if there are disturbing forces of the same frequency as the critical speed for the condition $\omega_1 = -\omega$, then rotation of the bent shaft in a direction opposite to that of the rotating disc may take place.*

Vibration of a Rigid Rotor with Flexible Bearings.—Equations (92) and (93) can be used also in the study of vibrations of a rigid rotor, having bearings in flexible pedestals (Fig. 167). Let y_1, z_1 and y_2, z_2 be small displacements of the bearings during vibration. Taking these displacements as coordinates of the oscillating rotor, the displacements of the center of gravity and the angular displacements of the axis of the rotor will be (see Fig. 167).

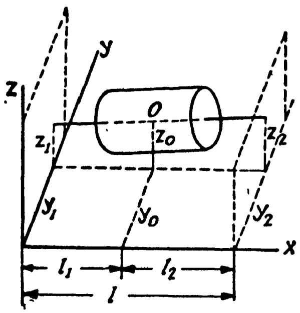

FIG. 167.

$$y_0 = y_1 \frac{l_2}{l} + y_2 \frac{l_1}{l},$$

$$z_0 = z_1 \frac{l_2}{l} + z_2 \frac{l_1}{l},$$

$$\beta = \frac{y_2 - y_1}{l},$$

$$\gamma = \frac{z_2 - z_1}{l}.$$

Let c_1, c_2, d_1 and d_2 be constants depending on the flexibility of the pedestals in the horizontal and vertical directions, such that $-c_1y_1$, $-c_2y_2$ are horizontal and $-d_1z_1$, $-d_2z_2$ are the vertical reactions of the bearings due to the small displacements y_1, y_2, z_1 and z_2 in the y and z directions. Then the equations of motion of the center of gravity (92) become

$$\begin{aligned} &\frac{W}{gl}(l_2\ddot{y}_1 + l_1\ddot{y}_2) + c_1y_1 + c_2y_2 = 0, \\ &\frac{W}{gl}(l_2\ddot{z}_1 + l_1\ddot{z}_2) + d_1z_1 + d_2z_2 = 0. \end{aligned} \tag{v}$$

The eqs. (93) representing the rotations of the rotor about the y and z axis will be in this case

$$\begin{aligned} &I\omega\frac{\dot{y}_2 - \dot{y}_1}{l} - I_1\frac{\ddot{z}_2 - \ddot{z}_1}{l} = z_2d_2l_2 - z_1d_1l_1, \\ &I\omega\frac{\dot{z}_2 - \dot{z}_1}{l} + I_1\frac{\ddot{y}_2 - \ddot{y}_1}{l} = -y_2c_2l_2 + y_1c_1l_1. \end{aligned} \tag{w}$$

* See A. Stodola, "Dampf- und Gasturbinen" (1924), p. 367.

The four equations (v) and (w) completely describe the free vibrations of a rigid rotor on flexible pedestals. Substituting in these equations

$$y_1 = A \sin pt; \qquad y_2 = B \sin pt; \qquad z_1 = C \cos pt; \qquad z_2 = D \cos pt,$$

four homogeneous linear equations in A, B, C, and D will be obtained. Equating the determinant of these equations to zero, we get the frequency equation from which the frequencies of the four natural modes of vibration of the rotor can be calculated.

Consider now a forced vibration of the rotor produced by some eccentrically attached mass. The effect of such an unbalance will be equivalent to the action of a disturbing force with the components

$$Y = A \cos \omega t; \qquad Z = B \sin \omega t,$$

applied to the center of gravity and to a couple with the components,

$$M_y = C \sin \omega t; \qquad M_z = D \cos \omega t.$$

Instead of the eqs. (v) and (w) we obtain

$$\begin{aligned}
&\frac{W}{gl} l_2\ddot{y}_1 + l_1\ddot{y}_2) + c_1y_1 + c_2y_2 = A \cos \omega t, \\
&\frac{W}{gl} (l_2\ddot{z}_1 + l_1\ddot{z}_2) + d_1z_1 + d_2z_2 = B \sin \omega t, \\
&I\omega \frac{\dot{y}_2 - \dot{y}_1}{l} - I_1 \frac{\ddot{z}_2 - \ddot{z}_1}{l} = z_2d_2l_2 - z_1d_1l_1 + C \sin \omega t, \\
&I\omega \frac{\dot{z}_2 - \dot{z}_1}{l} + I_1 \frac{\ddot{y}_2 - \ddot{y}_1}{l} = - y_2c_2l_2 + y_1c_1l_1 + D \cos \omega t.
\end{aligned} \tag{a'}$$

The particular solution of these equations representing the forced vibration of the rotor will be of the form

$$y_1 = A' \cos \omega t; \quad y_2 = B' \cos \omega t; \quad z_1 = C' \sin \omega t; \quad z_2 = D' \sin \omega t.$$

Substituting in eqs. (a)', the amplitude of the forced vibration will be found. During this vibration the axis of the rotor describes a surface given by the equations

$$y = (a + bx) \cos \omega t,$$

$$z = (c + dx) \sin \omega t,$$

in which a, b, c and d are constants. We see that every point of the axis describes an ellipse given by the equation,

$$\frac{y^2}{(a + bx)^2} + \frac{z^2}{(c + dx)^2} = 1.$$

For two points of the axis, namely, for

$$x_1 = -\frac{a}{b} \qquad \text{and} \qquad x_2 = -\frac{c}{d},$$

the ellipses reduce to straight lines and the general shape of the surface described by the axis of the rotor will be as shown in Fig. 168. It is seen

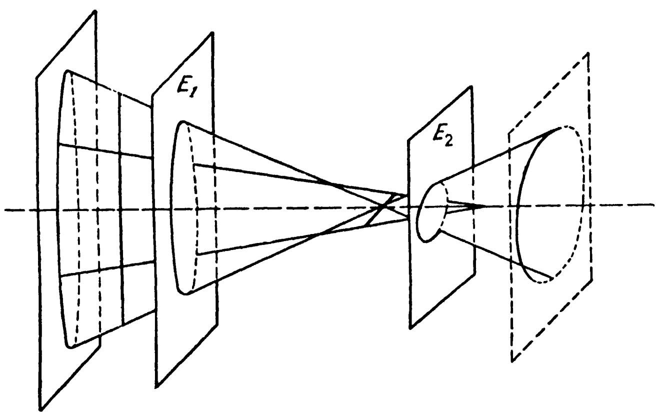

FIG. 168.

that the displacements of a point on the axis of the rotor depend not only upon the magnitude of the disturbing force (amount of unbalance) but also upon the position of the point along the axis and on the direction in which the displacement is measured.

In the general case the unbalance can be represented by two eccentrically attached masses (see Art. 13) and the forced vibrations of the rotor can be obtained by superimposing two vibrations of such kind as considered above and having a certain difference in phase.* From the linearity of the equations (a') it can also be concluded that by putting correction

* This question is discussed in detail in the paper by V. Blaess, "Über den Massenausgleich raschumlaufender Körper," Z. f. angewandte Mathematik und Mechanik, Vol. 6 (1926), p. 429. See also paper by D. M. Smith, l. c. page 213.

weights in two planes the unbalance always can be removed; it is only necessary to determine the correction weights in such a manner that the corresponding centrifugal forces will be in equilibrium with the disturbing forces due to unbalance.*

49. Effect of Weight of Shaft and Discs on the Critical Speed.—In our previous discussion the effect of the weight of the rotating discs was excluded by assuming that the axis of the shaft is vertical. In the case of horizontal shafts the weights of the discs must be considered as disturbing forces which at a certain speed produce considerable vibration in the shaft. This speed is usually called "critical speed of the second order."† For determining this speed a more detailed study of the motion of discs is necessary. In the following the simplest case of a single disc will be considered and it will be assumed that the disc is attached to the shaft at the cross section in which the tangent to the deflection curve of the shaft remains parallel to the center line of the bearings. In this manner the "gyroscopic effect," discussed in the previous article, will be excluded and only the motion of the discs in its own plane needs to be considered. Let us begin with the case when the shaft is vertical. Then xy represents the horizontal plane of the disc and O the center of the vertical shaft in its undeflected position (see Fig. 169). During the vibration let S be the instantaneous position of the center of the shaft and C, the instantaneous position of the center of gravity of the disc so that $CS = e$ represents the eccentricity with which the disc is attached to the shaft. Other notations will be as follows:

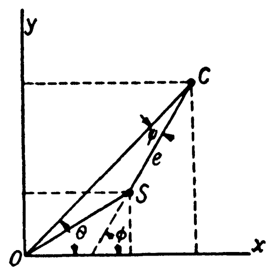

FIG. 169.

m = the mass of the disc.

mi^2 = moment of inertia of the disc about the axis through C and perpendicular to the disc.

k = spring constant of the shaft equal to the force in the xy plane necessary to produce unit deflection in this plane.

$\omega_{cr} = \sqrt{k/m}$ = the critical speed of the first order (see article 17).

x, y = coordinates of the center of gravity C of the disc during motion.

* The effect of flexibility of the shaft will be considered later (see Art. 50).

† A. Stodola was the first to discuss this problem. The literature on the subject can be found in his book, 6th Ed., p. 929. See also the paper by T. Pöschl in Zeitschr. f. angew. Mathem. u. Mech., Vol. 3 (1923), p. 297.

φ = the angle of rotation of the disc equal to the angle between the radius SC and x axis.
θ = the angle of rotation of the vertical plane OC.
ψ = the angle of rotation of the disc with respect to the plane OC.

Then $\varphi = \psi + \theta$. The coordinates x and y of the center of gravity C and the angle of rotation φ will be taken as coordinates determining the position of the disc in the plane xy.

The differential equations of motion of the center of gravity C can easily be written in the usual way if we note that only one force, the elastic reaction of the shaft, is acting on the disc in the xy plane. This force is proportional to the deflection OS of the shaft and its components in the x and y directions, proportional to the coordinates of the point S, will be $-k(x - e\cos\varphi)$ and $-k(y - e\sin\varphi)$ respectively. Then the differential equations of motion of the center C will be

$$m\ddot{x} = -k(x - e\cos\varphi); \qquad m\ddot{y} = -k(y - e\sin\varphi)$$

or

$$\begin{aligned} m\ddot{x} + kx &= ke\cos\varphi, \\ m\ddot{y} + ky &= ke\sin\varphi. \end{aligned} \tag{a}$$

The third equation will be obtained by using the principle of angular momentum. The angular momentum of the disc about the O axis consists (1) of the angular momentum $mi^2\dot{\varphi}$ of the disc rotating with the angular velocity $\dot{\varphi}$ about its center of gravity and (2) of the angular momentum $m(x\dot{y} - y\dot{x})$ of the mass m of the disc concentrated at its center of gravity. Then the principle of angular momentum gives the equation

$$\frac{d}{dt}\left\{mi^2\dot{\varphi} + m(x\dot{y} - y\dot{x})\right\} = M,$$

or

$$mi^2\ddot{\varphi} + m(x\ddot{y} - y\ddot{x}) = M, \tag{b}$$

in which M is the torque transmitted to the disc by the shaft.

The equations (a) and (b) completely describe the motion of the disc. When $M = 0$ a particular solution of the equations (a) and (b) will be obtained by assuming that the center of gravity C of the disc remains in the plane OS of the deflection curve of the shaft and describes while rotating at constant angular velocity $\dot{\varphi} = \omega$, a circle of radius r. Then substituting in equations (a) $x = r\cos\omega t$; $y = r\sin\omega t$ and taking $\varphi = \omega t$ for the case represented in Fig. 170*a*, and $\varphi = \omega t + \pi$ for the case represented in Fig. 170*b*, we obtain

$$r_0 = \frac{ke}{k - m\omega^2} = \frac{e\omega_{cr}^2}{\omega_{cr}^2 - \omega^2} \quad \text{for} \quad \omega < \omega_{cr},$$

$$r_0 = -\frac{ke}{k - m\omega^2} = \frac{e\omega_{cr}^2}{\omega^2 - \omega_{cr}^2} \quad \text{for} \quad \omega > \omega_{cr}.$$

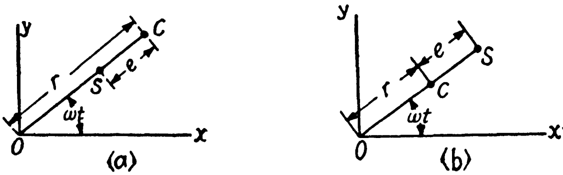

FIG. 170.

These results coincide with those obtained before from elementary considerations (see Art. 17).

Let us now consider the case when the torque M is different from zero and such that *

$$M = m(x\ddot{y} - y\ddot{x}). \tag{c}$$

Then from eq. (*b*) we conclude that

$$\dot{\varphi} = \omega = \text{const.},$$

and by integrating we obtain

$$\varphi = \omega t + \varphi_0, \tag{d}$$

in which φ_0 is an arbitrary constant determining the initial magnitude of the angle φ.

Substituting (*d*) in eqs. (*a*) and using the notation $\omega_{cr}^2 = k/m$, we obtain

$$\ddot{x} + \omega_{cr}^2 x = \omega_{cr}^2 e \cos(\omega t + \varphi_0),$$

$$\ddot{y} + \omega_{cr}^2 y = \omega_{cr}^2 e \sin(\omega t + \varphi_0). \tag{e}$$

* This case is discussed in detail in the dissertation "Die kritischen Zustände zweiter Art rasch umlaufender Wellen," by P. Schröder, Stuttgart, 1924. This paper contains very complete references to the new literature on the subject.

It is easy to show by substitution that

$$
\begin{aligned}
x &= -\frac{M_1}{ek}\cos(\omega_{cr}t + \gamma_1 + \varphi_0) + \frac{e\omega_{cr}^2}{\omega_{cr}^2 - \omega^2}\cos(\omega t + \varphi_0),\\
y &= -\frac{M_1}{ek}\sin(\omega_{cr}t + \gamma_1 + \varphi_0) + \frac{e\omega_{cr}^2}{\omega_{cr}^2 - \omega^2}\sin(\omega t + \varphi_0),
\end{aligned} \tag{f}
$$

represent a solution of the eqs. (*e*).

Substituting (*f*) in eq. (*c*) we obtain

$$
M = M_1 \sin\{(\omega_{cr} - \omega)t + \gamma_1\}. \tag{g}
$$

It can be concluded that under the action of the *pulsating moment* (*g*) the disc is rotating with a constant angular velocity and at the same time its center of gravity performs a combined oscillatory motion represented by the eqs. (*f*).

In the same manner it can be shown that under the action of a pulsating torque

$$
M = M_2 \sin\{(\omega_{cr} + \omega)t + \gamma_2\},
$$

the disc also rotates with a constant speed ω and its center performs oscillatory motions given by the equations

$$
\begin{aligned}
x &= \frac{M_2}{ek}\cos(\omega_{cr}t + \gamma_2 - \varphi_0) + \frac{e\omega_{cr}^2}{\omega_{cr}^2 - \omega^2}\cos(\omega t + \varphi_0),\\
y &= -\frac{M_2}{ek}\sin(\omega_{cr}t + \gamma_2 - \varphi_0) + \frac{e\omega_{cr}^2}{\omega_{cr}^2 - \omega^2}\sin(\omega t + \varphi_0).
\end{aligned} \tag{h}
$$

Combining the solutions (*f*) and (*h*) the complete solution of the eqs. (*e*), containing four arbitrary constants M_1, M_2, γ_1 and γ_2 will be obtained. This result can now be used for explaining the vibrations produced by the weight of the disc itself.

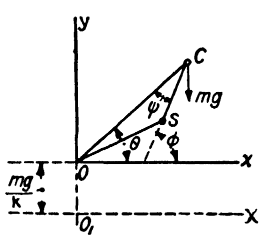

FIG. 171.

Assume that the shaft is in a horizontal position and the y axis is upwards, then by adding the weight of the disc we will obtain Fig. 171, instead of Fig. 169. The equations (*a*) and (*b*) will be replaced in this case by the following system of equations:

$$
\begin{aligned}
m\ddot{x} + kx &= ke\cos\varphi,\\
m\ddot{y} + ky &= ke\sin\varphi - mg,\\
mi^2\ddot{\varphi} + m(x\ddot{y} - y\ddot{x}) &= M - mgx.
\end{aligned} \tag{k}
$$

Let us displace the origin of coordinates from O to O_1 as shown in the figure; then by letting

$$y_1 = y + \frac{mg}{k},$$

eqs. (k) can be represented in the following form:

$$\begin{aligned} m\ddot{x} + kx &= ke \cos \varphi, \\ m\ddot{y}_1 + ky_1 &= ke \sin \varphi, \\ mi^2\ddot{\varphi} + m(x\ddot{y}_1 - y_1\ddot{x}) &= M - mge \cos \varphi. \end{aligned} \tag{l}$$

This system of equations coincides with the system of eqs. (a) and (b) and the effect of the disc's weight is represented by the pulsating torque $-\, mge \cos \varphi$. Imagine now that $M = 0$ and that the shaft is rotating with a constant angular velocity $\omega = \frac{1}{2}\omega_{cr}$. Then the effect of the weight of the disc can be represented in the following form

$$\begin{aligned} -mge \cos \varphi = -\, mge \cos (\omega t) &= mge \sin (\omega t - \pi/2) \\ &= mge \sin \left\{ (\omega_{cr} - \omega)t - \pi/2 \right\}. \end{aligned} \tag{m}$$

This disturbing moment has exactly the same form as the pulsating moment given by eq. (g) and it can be concluded that at the speed $\omega = \frac{1}{2}\omega_{cr}$, the pulsating moment due to the weight of the disc will produce vibrations of the shaft given by the equations (f). This is the so-called *critical speed of the second order*, which in many actual cases has been observed.* It should be noted, however, that vibrations of the same frequency can be produced also by variable flexibility of the shaft (see p. 154) and it is quite possible that in some cases where a critical speed of the second order has been observed the vibrations were produced by this latter cause.

50. Effect of Flexibility of Shafts on the Balancing of Machines.—In our previous discussion on the balancing of machines (see Art. 13) it was assumed that the rotor was an absolutely rigid body. In such a case complete balancing may be accomplished by putting correction weights in two arbitrarily chosen planes. The assumption neglecting the flexibility of the shaft is accurate enough at low speeds but for high speed machines and especially in the cases of machines working above the critical speed the deflection of the shaft may have a considerable effect and as a result of this, the rotor can be balanced only for one definite speed or at certain conditions cannot be balanced at all and will always give vibration troubles.

* See, O. Föppl, V.D.I., Vol. 63 (1919), p. 867.

The effect of the flexibility of the shaft will now be explained on a simple example of a shaft supported at the ends and carrying two discs (see Fig. 172). The deflection of the shaft y_1 under a load W_1 will depend not only on the magnitude of this load, but also on the magnitude of the load W_2. The same conclusion holds also for the deflection y_2 under the load W_2. By using the equations of the deflection curve of a shaft on two supports, the following expressions for the deflections can be obtained:

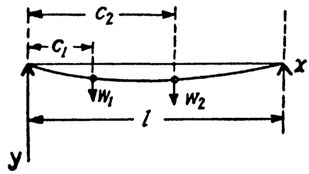

FIG. 172.

$$y_1 = a_{11}W_1 + a_{12}W_2,$$
$$y_2 = a_{21}W_1 + a_{22}W_2, \tag{a}$$

in which a_{11}, a_{12}, a_{21} and a_{22} remain constant for a given shaft and a given position of loads. These equations can be used now in calculating the deflections produced in the shaft by the centrifugal forces due to eccentricities of the discs.

Let m_1, m_2 = masses of discs I and II,
ω = angular velocity,
y_1, y_2 = deflections at the discs I and II, respectively,
c_1, c_2 = distances from the left support to the discs I and II,
Y_1, Y_2 = centrifugal forces acting on the shaft.

Assuming that only disc I has a certain eccentricity e_1 and taking the deflection in the plane of this eccentricity, the centrifugal forces acting on the shaft will be

$$Y_1 = (e_1 + y_1)m_1\omega^2; \qquad Y_2 = y_2m_2\omega^2,$$

or, by using equations similiar to eqs. (a), we obtain

$$Y_1 = e_1m_1\omega^2 + m_1\omega^2(a_{11}Y_1 + a_{12}Y_2),$$
$$Y_2 = m_2\omega^2(a_{21}Y_1 + a_{22}Y_2),$$

from which

$$Y_1 = \frac{e_1m_1\omega^2(1 - a_{22}m_2\omega^2)}{(1 - a_{11}m_1\omega^2)(1 - a_{22}m_2\omega^2) - m_1m_2a_{12}a_{21}\omega^4},$$
$$Y_2 = \frac{e_1a_{21}m_1m_2\omega^4}{(1 - a_{11}m_1\omega^2)(1 - a_{22}m_2\omega^2) - m_1m_2a_{12}a_{21}\omega^4}. \tag{b}$$

It is seen that instead of a centrifugal force $e_1m_1\omega^2$, which we have in the case of a rigid shaft, two forces Y_1 and Y_2 are acting on the flexible shaft. The unbalance will be the same as in the case of a rigid shaft on which a force $R_1 = Y_1 + Y_2$ is acting at the distance from the left support equal to

$$l_1 = \frac{Y_1 c_1 + Y_2 c_2}{Y_1 + Y_2}. \tag{c}$$

It may be seen from eqs. (*b*) that l_1 does not depend on the amount of eccentricity e_1, but only on the elastic properties of the shaft, the position and magnitude of the masses m_1 and m_2 and on the speed ω of the machine.

In the same manner as above the effect of eccentricity in disc II can be discussed and the result of eccentricities in both discs can be obtained by the principle of superposition. From this it can be concluded that at a given speed the unbalance in two discs on a flexible shaft is dynamically equivalent to unbalances in two definite planes of a rigid shaft. The position of these planes can be determined by using eq. (*c*) for one of the planes and an analogous equation for the second plane.

Similar conclusions can be made for a flexible shaft with any number n of discs * and it can be shown that the unbalance in these discs is equivalent to the unbalance in n definite planes of a rigid shaft. These planes remaining fixed at a given speed of the shaft, the balancing can be accomplished by putting correction weights in two planes arbitrarily chosen. At any other speed the planes of unbalance in the equivalent rigid shaft change their position and the rotor goes out of balance. This gives us an explanation why a rotor perfectly balanced in a balancing machine at a comparatively low speed may become out of balance at service speed. Thus balancing in the field under actual conditions becomes necessary. The displacements of the planes of unbalance with variation in speed is shown below for two particular cases. In Fig. 173 a shaft carrying three discs is represented. The changes with the speed in the distances l_1, l_2, l_3 of the planes of unbalance in the equivalent rigid shaft are shown in the figure by the curves l_1, l_2, l_3. It is seen that with an increase in speed these curves first approach each other, then go through a common point of intersection at the critical speed and above it diverge again. Excluding the region near the critical speed, the rotor can be balanced at any other speed, by putting correction weights in any two of the three discs. More difficult conditions are shown in Fig. 174. It is seen that at a speed equal to about 2150 r.p.m.

* A general investigation of the effect of flexibility of the shaft on the balancing can be found in the paper by V. Blaess, mentioned before (see p. 298). From this paper the figures 173 and 174 have been taken.

the curves l_1 and l_3 go through the same point A. The two planes of the equivalent rigid shaft coincide and it becomes impossible to balance the machine by putting correction weights in the discs 1 and 3. Practically

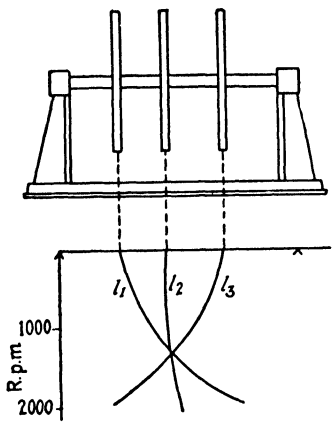

FIG. 173.

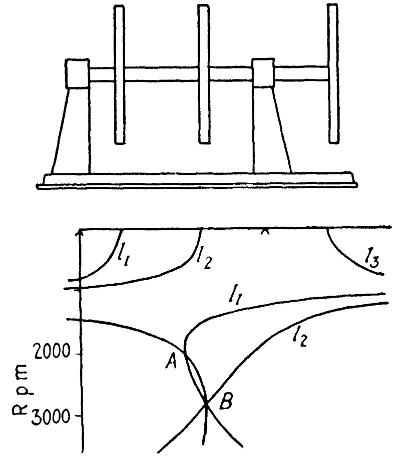

FIG. 174.

in a considerable region near the point A the conditions will be such that it will be difficult to obtain satisfactory balancing and heavy vibration troubles should be expected.

CHAPTER VI

VIBRATIONS OF ELASTIC BODIES

In considering the vibrations of elastic bodies it will be assumed that the material of the body is homogeneous, isotropic and that it follows Hooke's law. The differential equations of motion established in the previous chapter for a system of particles will also be used here.

In the case of elastic bodies, however, instead of several concentrated masses, we have a system consisting of an infinitely large number of particles between which elastic forces are acting. This system requires an infinitely large number of coordinates for specifying its position and it therefore has an infinite number of degrees of freedom because any small displacement satisfying the condition of continuity, i.e., a displacement which will not produce cracks in the body, can be taken as a possible or virtual displacement. On this basis it is seen that any elastic body can have an infinite number of natural modes of vibration.

In the case of thin bars and plates the problem of vibration can be considerably simplified. These problems, which are of great importance in many engineering applications, will be discussed in more detail * in the following chapter.

51. Longitudinal Vibrations of Prismatical Bars.—*Differential Equation of Longitudinal Vibrations.*—The following consideration is based on the assumption that during longitudinal vibration of a prismatical bar the cross sections of the bar remain plane and the particles in these cross sections perform only motion in an axial direction of the bar. The longitudinal extensions and compressions which take place during such a vibration of the bar will certainly be accompanied by some lateral deformation, but in the following only those cases will be considered where the length of the longitudinal waves is large in comparison with the cross sec-

* The most complete discussion of the vibration problems of elastic systems can be found in the famous book by Lord Rayleigh "Theory of Sound." See also H. Lamb, "The Dynamical Theory of Sound." A. E. H. Love, "Mathematical Theory of Elasticity," 4 ed. (1927), Handbuch der Physik, Vol. VI (1928), and Barré de Saint-Venant, Théorie de l'élasticité des corps solides. Paris, 1883.

tional dimensions of the bar. In these cases the lateral displacements during longitudinal vibration can be neglected without substantial errors.* Under these conditions the differential equation of motion of an element of the bar between two adjacent cross sections mn and m_1n_1 (see Fig. 175) may be written in the same manner as for a particle.

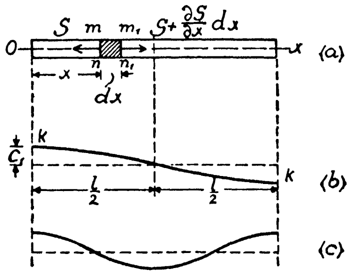

FIG. 175.

Let u = the longitudinal displacement of any cross section mn of the bar during vibration,
e = unit elongation,
E = modulus of elasticity,
A = cross sectional area,
$S = AEe$ = longitudinal tensile force,
γ = weight of the material of the bar per unit volume,
l = the length of the bar.

Then the unit elongation and the tensile force at any cross section mn of the bar will be

$$e = \frac{\partial u}{\partial x}; \qquad S = AE\frac{\partial u}{\partial x}.$$

For an adjacent cross section the tensile force will be

$$S + dS = AE\left(\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}dx\right).$$

Taking into consideration that the inertia force of the element mnm_1n_1 of the bar is

$$-\frac{A\gamma dx}{g}\frac{\partial^2 u}{\partial t^2},$$

and using the D'Alembert's principle, the following differential equation of motion of the element mnm_1n_1 will be obtained

* A complete solution of the problem on longitudinal vibrations of a cyclindrical bar of circular cross section, in which the lateral displacements are also taken into consideration, was given by L. Pochhammer, Jr. f. Mathem., Vol. 81 (1876), p. 324. See also E. Giebe u. E. Blechschmidt, Annalen d. Phys. 5 Folge, Vol. 18, p. 457, 1933.

$$-\frac{A\gamma}{g}\frac{\partial^2 u}{\partial t^2} + AE\frac{\partial^2 u}{\partial x^2} = 0,$$

or

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \tag{94}$$

in which *

$$a^2 = \frac{Eg}{\gamma}. \tag{95}$$

Solution by Trigonometric Series.—The displacement u, depending on the coordinate x and on the time t, should be such a function of x and t as to satisfy the partial differential eq. (94). Particular solutions of this equation can easily be found by taking into consideration 1, that in the general case any vibration of a system can be resolved into the natural modes of vibration and 2, when a system performs one of its natural modes of vibration all points of the system execute a simple harmonic vibration and keep step with one another so that they pass simultaneously through their equilibrium positions. Assume now that the bar performs a natural mode of vibration, the frequency of which is $p/2\pi$, then the solution of eq. (94) should be taken in the following form:

$$u = X(A \cos pt + B \sin pt), \tag{a}$$

in which A and B are arbitrary constants and X a certain function of x alone, determining the shape of the normal mode of vibration under consideration, and called "*normal function.*" This function should be determined in every particular case so as to satisfy the conditions at the ends of the bar. As an example consider now the longitudinal vibrations of a bar with free ends. In this case the tensile force at the ends during vibration should be equal to zero and we obtain the following end conditions (see Fig. 175)

$$\left(\frac{\partial u}{\partial x}\right)_{x=0} = 0; \qquad \left(\frac{\partial u}{\partial x}\right)_{x=l} = 0. \tag{b}$$

Substituting (a) in eq. (94) we obtain

$$-p^2 X = a^2 \frac{d^2 X}{dx^2},$$

from which

$$X = C \cos\frac{px}{a} + D \sin\frac{px}{a}. \tag{c}$$

* It can be shown that a is the velocity of propagation of waves along the bar.

In order to satisfy the first of the conditions (*b*) it is necessary to put $D = 0$. The second of the conditions (*b*) will be satisfied when

$$\sin \frac{pl}{a} = 0. \tag{96}$$

This is the "*frequency equation*" for the case under consideration from which the frequencies of the natural modes of the longitudinal vibrations of a bar with free ends can be calculated. This equation will be satisfied by putting

$$\frac{pl}{a} = i\pi, \tag{d}$$

where i is an integer. Taking $i = 1, 2, 3, \cdots$, the frequencies of the various modes of vibration will be obtained. The frequency of the fundamental type of vibration will be found by putting $i = 1$, then

$$p_1 = \frac{a\pi}{l} = \frac{\pi}{l}\sqrt{\frac{Eg}{\gamma}}. \tag{97}$$

The corresponding period of vibration will be

$$\tau_1 = \frac{2\pi}{p_1} = 2l\sqrt{\frac{\gamma}{Eg}}. \tag{98}$$

The shape of this mode of vibration, obtained from eq. (*c*), is represented in Fig. 175*b*, by the curve *kk*, the ordinates of which are equal to

$$X_1 = C_1 \cos \frac{p_1 x}{a} = C_1 \cos \frac{\pi x}{l}.$$

In Fig. 175*c*, the second mode of vibration is represented in which

$$\frac{p_2 l}{a} = 2\pi; \qquad \text{and} \qquad X_2 = C_2 \cos \frac{2\pi x}{l}.$$

The general form of a particular solution (*a*) of eq. (94) will be

$$u = \cos \frac{i\pi x}{l}\left(A_i \cos \frac{i\pi a t}{l} + B_i \sin \frac{i\pi a t}{l}\right). \tag{e}$$

By superimposing such particular solutions any longitudinal vibration of the bar * can be represented in the following form:

$$u = \sum_{i=1,2,3,\cdots}^{i=\infty} \cos\frac{i\pi x}{l}\left(A_i \cos\frac{i\pi at}{l} + B_i \sin\frac{i\pi at}{l}\right). \tag{99}$$

The arbitrary constants A_i, B_i always can be chosen in such a manner as to satisfy any initial conditions.

Take, for instance, that at the intial moment $t = 0$, the displacements u are given by the equation $(u)_{t=0} = f(x)$ and the initial velocities by the equation $(\dot{u})_{t=0} = f_1(x)$. Substituting $t = 0$ in eq. (99), we obtain

$$f(x) = \sum_{i=1}^{i=\infty} A_i \cos\frac{i\pi x}{l}. \tag{f}$$

By substituting $t = 0$ in the derivative with respect to t of eq. (99), we obtain

$$f_1(x) = \sum_{i=1}^{i=\infty} \frac{i\pi a}{l} B_i \cos\frac{i\pi x}{l}. \tag{g}$$

The coefficients A_i and B_i in eqs. (f) and (g) can now be calculated, as explained before (see Art. 18) by using the formulae:

$$A_i = \frac{2}{l}\int_0^l f(x)\cos\frac{i\pi x}{l}\,dx, \tag{h}$$

$$B_i = \frac{2}{i\pi a}\int_0^l f_1(x)\cos\frac{i\pi x}{l}\,dx. \tag{k}$$

As an example, consider now the case when a prismatical bar compressed by forces applied at the ends, is suddenly released of this compression at the initial moment $t = 0$. By taking †

$$(u)_{t=0} = f(x) = \frac{el}{2} - ex; \qquad f_1(x) = 0,$$

where e denotes the unit compression at the moment $t = 0$, we obtain from eqs. (h) and (k)

$$A_i = \frac{4el}{\pi^2 i^2} \text{ for } i = \text{odd}; \qquad A_i = 0 \text{ for } i = \text{even}; \qquad B_i = 0,$$

* Displacement of the bar as a rigid body is not considered here. An example where this displacement must be taken into consideration will be discussed on p. 316.

† It is assumed that the middle of the bar is stationary.

and the general solution (99) becomes

$$u = \frac{4el}{\pi^2} \sum_{i=1,3,5,\cdots}^{i=\infty} \frac{\cos \frac{i\pi x}{l} \cos \frac{i\pi at}{l}}{i^2}.$$

Only odd integers $i = 1, 3, 5, \cdots$ enter in this solution and the vibration is symmetrical about the middle cross section of the bar.

On the general solution 99 representing the vibration of the bar any longitudinal displacement of the bar as a rigid body can be superimposed.

Solution by using Generalized Coordinates.—Taking as generalized coordinates in this case the expressions in the brackets in eq. (*e*) and using the symbols q_i for these coordinates, we obtain

$$u = \sum_{i=1}^{i=\infty} q_i \cos \frac{i\pi x}{l}. \tag{l}$$

The potential energy of the system consisting in this case of the energy of tension and compression will be,

$$V = \frac{AE}{2} \int_0^l \left(\frac{\partial u}{\partial x}\right)^2 dx = \frac{AE\pi^2}{2l^2} \int_0^l \left(\sum_{i=1}^{i=\infty} iq_i \sin \frac{i\pi x}{l}\right)^2 dx = \frac{AE\pi^2}{4l} \sum_{i=1}^{i=\infty} i^2 q_i^2 \cdots. \tag{m}$$

In calculating the integral

$$\int_0^l \left(\Sigma\, iq_i \sin \frac{i\pi x}{l}\right)^2 dx$$

only the terms containing the squares of the coordinates q_i give integrals different from zero (see Art. 18).

The kinetic energy at the same time will be,

$$T = \frac{A\gamma}{2g} \int_0^l \left(\frac{\partial u}{\partial t}\right)^2 dx = \frac{A\gamma l}{4g} \sum_{i=1}^{i=\infty} \dot{q}_i^2. \tag{n}$$

Substituting T and V in Lagrange's eqs. (73) we obtain for each coordinate q_i the following differential equation

$$\ddot{q}_i + \frac{a^2\pi^2 i^2}{l^2} q_i = 0, \tag{p}$$

from which

$$q_i = A_i \cos \frac{i\pi at}{l} + B_i \sin \frac{i\pi at}{l}.$$

This result coincides completely with what was obtained before (see eq. e). We see that the equations (p) contain each only one coordinate q_i. The chosen coordinates are independent of each other and the corresponding vibrations are "principal" modes of vibration of the bar (see p. 197).

The application of generalized coordinates is especially useful in the discussion of forced vibrations. As an example, let us consider here the case of a bar with one end built in and another end free. The solution for this case can be obtained at once from expression (99). It is only necessary to assume in the previous case that the bar with free ends performs vibrations symmetrical about the middle of the bar. This condition will be satisfied by taking $i = 1, 3, 5 \cdots$ in solution (99). Then the middle section can be considered as fixed and each half of the bar will be exactly in the same condition as a bar with one end fixed and another free. Denoting by l the length of such a bar and putting the origin of coordinates at the fixed end, the solution for this case will be obtained by substituting $2l$ for l and $\sin i\pi x/2l$ for $\cos i\pi x/l$ in eq. (99). In this manner we obtain

$$u = \sum_{i=1,3,5,\cdots}^{i=\infty} \sin\frac{i\pi x}{2l}\left(A_i \cos\frac{i\pi at}{2l} + B_i \sin\frac{i\pi at}{2l}\right). \tag{100}$$

Now, if we consider the expressions in the brackets of the above solution as generalized coordinates and use the symbols q_i for them, we obtain,

$$u = \sum_{i=1,3,5,\cdots}^{i=\infty} q_i \sin\frac{i\pi x}{2l}. \tag{q}$$

Substituting this in the expressions for the potential and kinetic energy we obtain:

$$V = \frac{AE}{2}\int_0^l \left(\frac{\partial u}{\partial x}\right)^2 dx = \frac{\pi^2 AE}{16l}\sum_{i=1,3,5,\cdots}^{i=\infty} i^2 q_i^2, \tag{101}$$

$$T = \frac{A\gamma}{2g}\int_0^l (\dot{u})^2 dx = \frac{A\gamma l}{4g}\sum_{i=1,3,5,\cdots}^{\infty} \dot{q}_i^2. \tag{102}$$

Lagrange's equation for free vibration corresponding to any coordinate q_i will be as follows:

$$\ddot{q}_i + \frac{a^2 i^2 \pi^2}{4l^2} q_i = 0,$$

from which

$$q_i = A_i \cos\frac{i\pi at}{2l} + B_i \sin\frac{i\pi at}{2l}.$$

This coincides with what we had before (see eq. (100)).

Forced Vibrations.—If disturbing forces are acting on the bar, Lagrange's eqs. (74) will be

$$\frac{A\gamma l}{2g}\ddot{q}_i + \frac{\pi^2 i^2 AE}{8l} q_i = Q_i$$

or

$$\ddot{q}_i + \frac{a^2\pi^2 i^2}{4l^2} q_i = \frac{2g}{A\gamma l} Q_i, \tag{r}$$

in which Q_i denotes the generalized force corresponding to the generalized coordinate q_i. In determining this force the general method explained before (see p. 187) will be used. We give an increase δq_i to the coordinate q_i. The corresponding displacement in the bar, as determined from (q), is

$$\delta u = \delta q_i \sin\frac{i\pi x}{2l}.$$

The work done by the disturbing forces on this displacement should now be calculated. This work divided by δq_i represents the generalized force Q_i. Substituting this in eq. (r), the general solution of this equation can easily be obtained, by adding to the free vibrations, obtained above, the vibrations produced by the disturbing force Q_i. This latter vibration is taken usually in the form of a definite integral.* Then,

$$q_i = A_i \cos\frac{i\pi at}{2l} + B_i \sin\frac{i\pi at}{2l} + \frac{4g}{A\gamma a\pi i}\int_0^t Q_i \sin\frac{i\pi a}{2l}(t - t_1)dt_1. \tag{s}$$

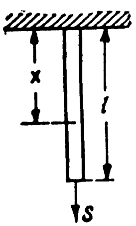

FIG. 176.

The first two terms in this solution represent a free vibration due to the initial displacement and initial impulse. The third represents the vibration produced by the disturbing force. Substituting solution (s) in eq. (q) the general expression for the vibrations of the bar will be obtained. As an example, the vibration produced by a force $S = f(t)$ acting on the free end of the bar (see Fig. 176) will now be considered. Giving an increase δq_i to the coordinate q_i; the corresponding displacement (see eq. q) will be

$$\delta u = \delta q_i \sin\frac{i\pi x}{2l}.$$

The work produced by the disturbing force on this displacement is

$$S\delta q_i \sin\frac{i\pi}{2}$$

* See eq. (48), p. 104.

and we obtain

$$Q_i = S \sin \frac{i\pi}{2} = (-1)^{\frac{i-1}{2}} S.$$

where $i = 1, 3, 5, \ldots$.

Substituting in (s) and taking into consideration only that part of the vibration, produced by the disturbing force, we obtain,

$$q_i = (-1)^{\frac{i-1}{2}} \frac{4g}{A\gamma a\pi i} \int_0^t S \sin \frac{i\pi a}{2l} (t - t_1)dt_1.$$

Substituting in (q) and considering the motion of the lower end of the bar ($x = l$) we have

$$(u)_{x=l} = \frac{4g}{A\gamma a\pi} \sum_{i=1,3,5,\ldots}^{i=\infty} \frac{1}{i} \int_0^t S \sin \frac{i\pi a}{2l} (t - t_1)dt_1. \tag{u}$$

In any particular case it is only necessary to substitute $S = f(t_1)$ in (u) and perform the integration indicated. Let us take, for instance, the particular case of the vibrations produced in the bar by a constant force suddenly applied at the initial moment ($t = 0$). Then, from (u), we obtain

$$(u)_{x=l} = \frac{8glS}{A\gamma a^2\pi^2} \sum_{i=1,3,5,\ldots}^{i=\infty} \frac{1}{i^2}\left(1 - \cos\frac{i\pi at}{2l}\right). \tag{103}$$

It is seen that all modes of vibration will be produced in this manner, the periods and frequencies of which are

$$\tau_i = \frac{4l}{ai}; \qquad f_i = \frac{1}{\tau_i} = \frac{ai}{4l}.$$

The maximum deflection will occur when $\cos(i\pi at/2l) = -1$. Then

$$(u)_{\substack{x=l \\ t=\frac{2l}{a}}} = \frac{16glS}{A\gamma a^2\pi^2} \sum_{i=1,3,5,\ldots}^{i=\infty} \frac{1}{i^2}$$

or by taking into consideration that

$$a^2 = \frac{Eg}{\gamma} \quad \text{and} \quad \sum_{i=1,3,5,\ldots}^{i=\infty} \frac{1}{i^2} = \frac{1}{1} + \frac{1}{3^2} + \frac{1}{5^2} \cdots = \frac{\pi^2}{8}$$

we obtain

$$(u)_{x=l} = \frac{2lS}{AE}.$$

We arrive in this manner at the well known conclusion that a suddenly applied force produces twice as great a deflection as one gradually applied.*

As another example let us consider the longitudinal vibration of a bar with free ends (Fig. 175) produced by a longitudinal force S suddenly applied at the end $x = l$. Superposing on the vibration of the bar given by eq. (l) a displacement q_0 of the bar as a rigid body the displacement u can be represented in the following form:

$$u = q_0 + q_1 \cos \frac{\pi x}{l} + q_2 \cos \frac{2\pi x}{l} + q_3 \cos \frac{3\pi x}{l} + \cdots. \tag{v}$$

The expressions for potential and kinetic energy, from (m) and (n) will be

$$V = \frac{AE\pi^2}{4l} \sum_{i=1}^{i=\infty} i^2 q_i^2; \qquad T = \frac{A\gamma l}{2g} \dot{q}_0^2 + \frac{A\gamma l}{4g} \sum_{i=1}^{i=\infty} \dot{q}_i^2$$

and the equations of motion become

$$\begin{array}{l} \dfrac{A\gamma l}{g} \ddot{q}_0 = Q_0, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \dfrac{A\gamma l}{2g} \ddot{q}_i + \dfrac{AE\pi^2 i^2}{2l} q_i = Q_i, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array} \tag{w}$$

By using the same method as before (see p. 314) it can be shown that in this case

$$Q_0 = S \quad \text{and} \quad Q_i = (-1)^i S.$$

Then assuming that the initial velocities and the initial displacements are equal to zero, we obtain, from eqs. (w):

$$q_0 = \frac{gSt^2}{2A\gamma l},$$

$$q_i = (-1)^i \frac{2g}{A\pi a\gamma i} \int_0^t S \sin \frac{i\pi a}{l} (t - t_1) dt_1 = \frac{(-1)^i 2glS}{A\pi^2 i^2 \gamma a^2} \left(1 - \cos \frac{i\pi at}{l}\right).$$

* For a more detailed discussion of this subject see the next article, p. 323.

Substituting in eq. (v), the following solution for the displacements produced by a suddenly applied force S will be obtained

$$u = \frac{gSt^2}{2A\gamma l} + \frac{2glS}{A\pi^2a^2\gamma}\sum_{i=1}^{i=\infty}\frac{(-1)^i}{i^2}\cos\frac{i\pi x}{l}\left(1 - \cos\frac{i\pi at}{l}\right).$$

The first term on the right side represents the displacement calculated as for a rigid body. To this displacement, vibrations of a bar with free ends are added. Using the notations $\delta = (Sl/AE)$ for the elongation of the bar uniformly stretched by the force S, and $\tau = (2l/a)$ for the period of the fundamental vibration, the displacement of the end $x = l$ of the bar will be

$$(u)_{x=l} = \frac{2\delta t^2}{\tau^2} + \frac{2\delta}{\pi^2}\sum_{i=1}^{i=\infty}\frac{1}{i^2}\left(1 - \cos\frac{2i\pi t}{\tau}\right).$$

The maximum displacement, due to vibration, will be obtained when $t = (\tau/2)$. Then

$$(u)_{x=l} = \frac{\delta}{2} + \frac{4\delta}{\pi^2}\left(\frac{1}{1} + \frac{1}{9} + \frac{1}{25} + \cdots\right) = \frac{\delta}{2} + \frac{4\delta}{\pi^2}\frac{\pi^2}{8} = \delta.$$

An analogous problem is encountered in investigating the vibrations produced during the lifting of a long drill stem as used in deep oil wells.

52. Vibration of a Bar with a Load at the End.—*Natural Vibrations.*—The problem of the vibration of a bar with a load at the end (Fig. 177) may have a practical application not only in the case of prismatical bars but also when the load is supported by a helical spring as in the case of an indicator spring (see p. 28). If the mass of the bar or of the spring be small in comparison with the mass of the load at the end it can be neglected and the problem will be reduced to that of a system with one degree of freedom (see Fig. 1). In the following the effect of the mass of the bar will be considered in detail.* Denoting the longitudinal displacements from the position of equilibrium by u and using the differential equation (94) of the longitudinal vibrations developed in the previous paragraph, we obtain

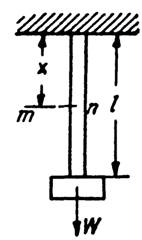

FIG. 177.

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}, \tag{94'}$$

where

$$a^2 = \frac{Eg}{\gamma}$$

* See author's paper, Bull. Polyt. Inst. Kiev, 1910, and Zeitschr. f. Math. u. Phys. V. 59, 1911. See also A. N. Kryloff, "Differential Eq. of Math. Phys.," p. 308, 1913, S. Petersburg.

for a prismatical bar, and

$$a^2 = \frac{klg}{w}$$

for a helical spring. In this latter case k is the spring constant, this being the load necessary to produce a total elongation of the spring equal to unity. l is the length of the spring and w is the weight of the spring per unit length. The end conditions will be as follows.

At the built-in end the displacement should be zero during vibration and we obtain

$$(u)_{x=0} = 0. \tag{a}$$

At the lower end, at which the load is attached, the tensile force in the bar must be equal to the inertia force of the oscillating load W and we have*

$$AE\left(\frac{\partial u}{\partial x}\right)_{x=l} = -\frac{W}{g}\left(\frac{\partial^2 u}{\partial t^2}\right)_{x=l}. \tag{b}$$

Assuming that the system performs one of the principal modes of vibration we obtain

$$u = X(A\cos pt + B\sin pt), \tag{c}$$

in which X is a *normal function* of x alone, determining the shape of the mode of vibration.

Substituting (c) in eq. (94′) we obtain

$$a^2\frac{d^2X}{dx^2} + p^2X = 0,$$

from which

$$X = C\cos\frac{px}{a} + D\sin\frac{px}{a}, \tag{d}$$

where C and D are constants of integration.

In order to satisfy condition (a) we have to take $C = 0$ in solution (d). From condition (b) we obtain

$$AE\frac{p}{a}\cos\frac{pl}{a} = \frac{W}{g}p^2\sin\frac{pl}{a}. \tag{b)^1$$

Let $\alpha = A\gamma l/W$ is ratio of the weight of the bar to the weight of the load W and $\beta = pl/a$. Then eq. $(b)^1$ becomes

$$\alpha = \beta\tan\beta. \tag{104}$$

This is the *frequency equation* for the case under consideration, the roots of which can be easily obtained graphically, provided the ratio α be known. The fundamental type of vibration is usually the most important in practical applications and the values β_1 of the smallest root of eq. (104) for various values of α are given in the table below.

$\alpha =$	.01	.10	.30	.50	.70	.90	1.00	1.50	2.00	3.00	4.00	5.00	10.0	20.0	100.0	∞
$\beta_1 =$	.10	.32	.52	.65	.75	.82	.86	.98	1.08	1.20	1.27	1.32	1.42	1.52	1.568	$\pi/2$

* The constant load W, being in equilibrium with the uniform tension of the bar in its position of equilibrium, will not affect the end condition.

If the weight of the bar is small in comparison with the load W, the quantity α and the root β_1 will be small and equation (104) can be simplified by putting $\tan \beta = \beta$, then

$$\beta^2 = \alpha = \frac{A\gamma l}{W},$$

and we obtain

$$\beta = \frac{pl}{a} = \sqrt{\frac{A\gamma l}{W}} \tag{e}$$

$$\text{and } p = \frac{a}{l}\sqrt{\frac{A\gamma l}{W}} = \sqrt{\frac{g}{\delta_{st}}}, \tag{f}$$

where $\delta_{st} = Wl/AE$ represents the statical elongation of the bar under the action of the load W.

This result coincides with the one obtained before for a system with one degree of freedom (see eq. 6, p. 3). A better approximation will be obtained by substituting $\tan \beta = \beta + \beta^3/3$ in eq. (104). Then

$$\beta(\beta + \beta^3/3) = \alpha,$$

or

$$\beta = \sqrt{\frac{\alpha}{1 + \beta^2/3}}. \tag{g}$$

Substituting the first approximation (e) for β in the right side of this equation, we obtain

$$\beta = \sqrt{\frac{\alpha}{1 + \alpha/3}} \quad \text{and} \quad p = \sqrt{\frac{g}{\delta_{st}(1 + \alpha/3)}}. \tag{h}$$

Comparing (h) with (f) it can be concluded that the better approximation is obtained by adding one third of the weight of the bar to the weight W of the load. This is the well-known approximate solution obtained before by using Rayleigh's method (see p. 85).

Comparing the approximate solution (h) with the data of the table above it can be concluded that for $\alpha = 1$ the error arising from the use of the approximate formula is less than 1% and in all cases when the weight of the bar is less than the weight of the load it is satisfactory for practical applications.

Assuming that for a given α the consecutive roots $\beta_1, \beta_2, \beta_3, \ldots$ of the frequency equation (104) are calculated, and substituting $\beta_i a/l$ for p in solution (c) we obtain,

$$u_i = \sin\frac{\beta_i x}{l}\left(A_i \cos\frac{\beta_i at}{l} + B_i \sin\frac{\beta_i at}{l}\right).$$

This solution represents a principal mode of vibration of the order i of our system. By superimposing such vibrations any vibration of the bar with a load at the end can be obtained in the form of a series,

$$u = \sum_{i=1}^{i=\infty} \sin\frac{\beta_i x}{l}\left(A_i \cos\frac{\beta_i at}{l} + B_i \sin\frac{\beta_i at}{l}\right), \tag{k}$$

the constants A_i and B_i of which should be determined from the initial conditions.

Assume, for instance, that the bar is at rest under the action of a tensile force S applied at the lower end and that at the initial moment $t = 0$ this force is suddenly removed. For this case all the coefficients B_i in eq. (k) should be taken equal to zero because the initial velocities are zero. The coefficients A_i should be determined in such a manner as to represent the initial configuration of the system. From the uniform extension of the bar at the initial moment we obtain

$$(u)_{t=0} = \frac{Sx}{AE}.$$

Equation (k), for $t = 0$, yields

$$(u)_{t=0} = \sum_{i=1}^{i=\infty} A_i \sin \frac{\beta_i x}{l}.$$

The coefficients A_i should be determined in such a manner as to satisfy the equation

$$\sum_{i=1}^{i=\infty} A_i \sin \frac{\beta_i x}{l} = \frac{Sx}{AE}. \tag{l}$$

In determining these coefficients we proceed exactly as was explained in Art. 18. In order to obtain any coefficient A_i both sides of the above equation should be multiplied with $\sin (\beta_i x/l)dx$ and integrated from $x = 0$ to $x = l$. By simple calculations we obtain

$$\int_0^l \sin^2 \frac{\beta_i x}{l} dx = \frac{l}{2}\left(1 - \frac{\sin 2\beta_i}{2\beta_i}\right)$$

$$\frac{S}{AE}\int_0^l x \sin \frac{\beta_i x}{l} dx = \frac{Sl^2}{AE}\left(-\frac{\cos \beta_i}{\beta_i} + \frac{\sin \beta_i}{\beta_i^2}\right),$$

and also, by taking into consideration eq. (104) for every integer $m \neq i$

$$\int_0^l \sin \frac{\beta_i x}{l} \sin \frac{\beta_m x}{l} dx = -\frac{W}{A\gamma} \sin \beta_i \sin \beta_m = -\frac{l}{\alpha} \sin \beta_i \sin \beta_m.$$

Then, from eq. (l)

$$\int_0^l \sin \frac{\beta_i x}{l} \sum_{i=1}^{i=\infty} A_i \sin \frac{\beta_i x}{l} dx = \frac{S}{AE}\int_0^l x \sin \frac{\beta_i x}{l} dx,$$

or

$$A_i \frac{l}{2}\left(1 - \frac{\sin 2\beta_i}{2\beta_i}\right) - \frac{l}{\alpha} \sin \beta_i \sum_{m=1,2,3,\cdots i-1, i+1,\cdots}^{m=\infty} A_m \sin \beta_m = \frac{Sl^2}{AE}\left(-\frac{\cos \beta_i}{\beta_i} + \frac{\sin \beta_i}{\beta_i^2}\right)$$

Remembering that, from eq. (k),

$$\sum_{i=1,2,3,\cdots (i-1),(i+1)\cdots}^{i=\infty} A_m \sin \beta_m = (u)_{\substack{x=l \\ t=0}} - A_i \sin \beta_i = \frac{Sl}{AE} - A_i \sin \beta_i,$$

we obtain

$$A_i \frac{l}{2}\left(1 - \frac{\sin 2\beta_i}{2\beta_i}\right) - \frac{l}{\alpha} \sin \beta_i \left(\frac{Sl}{AE} - A_i \sin \beta_i\right) = \frac{Sl^2}{AE}\left(-\frac{\cos \beta_i}{\beta_i} + \frac{\sin \beta_i}{\beta_i^2}\right)$$

from which, by taking into consideration that (from eq. 104)

$$\frac{l}{\alpha}\sin\beta_i = \frac{l\cos\beta_i}{\beta_i},$$

we obtain

$$A_i = \frac{4Sl\sin\beta_i}{AE\beta_i(2\beta_i + \sin 2\beta_i)},$$

the initial displacement will be

$$(u)_{t=0} = \frac{Sx}{AE} = \frac{4Sl}{AE}\sum_{i=1}^{i=\infty}\frac{\sin\beta_i\sin\frac{\beta_i x}{l}}{\beta_i(2\beta_i + \sin 2\beta_i)}, \tag{105}$$

and the vibration of the bar will be represented in this case by the following series:

$$u = \frac{4Sl}{AE}\sum_{i=1}^{i=\infty}\frac{\sin\beta_i\sin\frac{\beta_i x}{l}\cos\frac{\beta_i at}{l}}{\beta_i(2\beta_i + \sin 2\beta_i)}. \tag{106}$$

Forced Vibrations.—In the following the forced vibrations of the system will be considered by taking the expressions in the brackets of eq. (k) for generalized coordinates. Then

$$u = \sum_{i=1}^{i=\infty} q_i \sin\frac{\beta_i x}{l}. \tag{m}$$

The potential energy of the system will be,

$$V = \frac{AE}{2}\int_0^l\left(\frac{\partial u}{\partial x}\right)^2 dx = \frac{AE}{2l^2}\int_0^l\left(\sum_{i=1}^{i=\infty}\beta_i q_i\cos\frac{\beta_i x}{l}\right)^2 dx.$$

It can be shown by simple calculations that, in virtue of eq. (104),

$$\int_0^l \cos\frac{\beta_n x}{l}\cos\frac{\beta_m x}{l}\,dx = 0 \quad \text{when} \quad m \neq n,^*$$

and

$$\int_0^l \cos^2\frac{\beta_m x}{l}\,dx = \frac{l}{2}\left(1 + \frac{\sin 2\beta_m}{2\beta_m}\right).$$

Substituting in the above expression for V we obtain,

$$V = \frac{AE}{4l}\sum_{i=1}^{i=\infty}\beta_i^2 q_i^2\left(1 + \frac{\sin 2\beta_i}{2\beta_i}\right). \tag{n}$$

The kinetic energy of the system will consist of two parts, the kinetic energy of the vibrating rod and the kinetic energy of the load at the end of the rod, and we obtain

$$T = \frac{A\gamma}{2g}\int_0^l (\dot{u})^2 dx + \frac{W}{2g}(\dot{u})^2_{x=l}.$$

* The same can be concluded also from the fact that the coordinates $q_1, q_2, \ldots$ are principal coordinates, hence the potential and kinetic energies should contain only squares of these coordinates.

Substituting (m) for the displacement u and performing the integrations:

$$\int_0^l \sin^2 \frac{\beta_m x}{l}\, dx = \frac{l}{2}\left(1 - \frac{\sin 2\beta_m}{2\beta_m}\right),$$

$$\int_0^l \sin \frac{\beta_m x}{l} \sin \frac{\beta_n x}{l}\, dx = -\frac{W}{A\gamma} \sin \beta_m \sin \beta_n, \quad \text{where} \quad m \neq n.$$

We obtain

$$T = \frac{A\gamma l}{4g} \sum_{i=1}^{i=\infty} \dot{q}_i^2 \left(1 - \frac{\sin 2\beta_i}{2\beta_i}\right) + \frac{W}{2g} \sum_{i=1}^{i=\infty} \dot{q}_i^2 \sin^2 \beta_i.$$

Now, from eq. (104), we have

$$\alpha = \frac{A\gamma l}{W} = \beta_i \tan \beta_i,$$

or

$$W = \frac{A\gamma l}{\beta_i \tan \beta_i}.$$

Substituting in the above expression for the kinetic energy we obtain,

$$T = \frac{A\gamma l}{4g} \sum_{i=1}^{i=\infty} \dot{q}_i^2 \left(1 + \frac{\sin 2\beta_i}{2\beta_i}\right). \tag{o}$$

It is seen that the expressions (n) and (o) for the potential and kinetic energy contain only squares of q_i and $\dot{q}_i$. The products of these quantities disappear because the terms of the series (k) and (l) are the *principal or natural modes* of vibration of the system under consideration and the coordinates q_i are the *principal coordinates* (see p. 197). Substituting (n) and (o) in Lagrange's equation (74) the following equation for any coordinate q_i will be obtained.

$$\frac{A\gamma l}{2g}\left(1 + \frac{\sin 2\beta_i}{2\beta_i}\right) \ddot{q}_i + \frac{AE}{2l} \beta_i^2 \left(1 + \frac{\sin 2\beta_i}{2\beta_i}\right) q_i = Q_i, \tag{p}$$

in which Q_i denotes the generalized force corresponding to the generalized coordinate q_i.

Considering only vibrations produced by a disturbing force and neglecting the free vibrations due to initial displacements and initial impulses, the solution of eq. (p) will be *

$$q_i = \frac{2g}{A\gamma l} \cdot \frac{l}{a\beta_i} \frac{2\beta_i}{2\beta_i + \sin 2\beta_i} \int_0^t Q_i \sin \frac{a\beta_i}{l} (t - t_1) dt_1,$$

where, as before,

$$a = \sqrt{\frac{Eg}{\gamma}}.$$

Substituting this into (m) the following general solution of the problem will be obtained:

$$u = \frac{4g}{Aa\gamma} \sum_{i=1}^{i=\infty} \frac{\sin \frac{\beta_i x}{l}}{2\beta_i + \sin 2\beta_i} \int_0^t Q_i \sin \frac{a\beta_i}{l} (t - t_1) dt_1. \tag{107}$$

* See eq. 48, p. 104.

In any particular case the corresponding value of Q_i should be substituted in this solution. By putting $x = l$ the displacements of the load W during vibration will be obtained.

Force Suddenly Applied.—Consider, as an example, the vibration produced by a constant force S suddenly applied at the lower end of the bar. The generalized force Q_i corresponding to any coordinate q_i in this case (see p. 314) will be

$$Q_i = S \sin \beta_i.$$

Substituting in eq. (107) we obtain for the displacements of the load W the following expression:

$$(u)_{x=l} = \frac{4gSl}{Aa^2\gamma} \sum_{i=1}^{i=\infty} \frac{\sin^2 \beta_i}{\beta_i(2\beta_i + \sin 2\beta_i)} \left(1 - \cos \frac{a\beta_i t}{l}\right). \tag{108}$$

Consider now the particular case when the load W at the end of the bar diminishes to zero and the conditions approach those considered in the previous article. In such a case α in eq. (104) becomes infinitely large and the roots of that transcendental equation will be

$$\beta_i = \frac{(2i - 1)\pi}{2}.$$

Substituting in eq. (108) the same result as in the previous article (see eq. 103, p. 315) will be obtained.

A second extreme case is when the load W is very large in comparison with the weight of the rod and α in eq. (104) approaches zero. The roots of this equation then approach the values:

$$\beta_i = (i - 1)\pi.$$

All terms in the series (108) except the first term, tend towards zero and the system approaches the case of one degree of freedom. The displacement of the lower end of the rod will be given in this case by the first term of (108) and will be

$$(u)_{x=l} = \frac{4gSl}{Aa^2\gamma} \frac{\sin^2 \beta_1}{\beta_1(2\beta_1 + \sin 2\beta_1)} \left(1 - \cos \frac{a\beta_1 t}{l}\right),$$

or by putting $\sin \beta_1 = \beta_1$ and $\sin 2\beta_1 = 2\beta_1$ we obtain

$$(u)_{x=l} = \frac{gSl}{Aa^2\gamma} \left(1 - \cos \frac{a\beta_1 t}{l}\right).$$

This becomes a maximum when

$$\cos \frac{a\beta_1 t}{l} = -1,$$

then

$$(u)_{\max} = \frac{2gSl}{Aa^2\gamma} = \frac{2Sl}{AE}.$$

This show that the maximum displacement produced by a suddenly applied force is twice as great as the static elongation produced by the same force.

This conclusion also holds for the case when $W = 0$ (see p. 316) but it will not be true in the general case given by eq. 108. To prove this it is necessary to observe that in the two particular cases mentioned above, the system at the end of a half period of the fundamental mode of vibration will be in a condition of instantaneous rest. At this moment the kinetic energy becomes equal to zero and the work done by the suddenly applied constant force is completely transformed into potential energy of deformation and it can be concluded from a statical consideration that the displacement of the point of application of the force should be twice as great as in the equilibrium configuration.

In the general case represented by eq. (108) the roots of eq. (104) are incommensurable and the system never passes into a configuration in which the energy is purely potential. Part of the energy always remains in the form of kinetic energy and the displacement of the point of application of the force will be less than twice that in the equilibrium configuration.

Comparison with Static Deflection.—The method of generalized coordinates, applied above, is especially useful for comparing the displacements of a system during vibration and the statical displacements which would be produced in the system if the disturbing forces vary very slowly. Such comparisons are necessary, for instance, in the study of steam and gas engine indicator diagrams, and of various devices used in recording gas pressures during explosions. The case of an indicator is represented by the scheme shown in Fig. 177. Assume that a pulsating force $S \sin \omega t$ is applied to the load W, representing the reduced mass of the piston (see p. 28). In order to find the generalized force in this case, the expression (m) for the displacements will be used. Giving to a coordinate q_i an increase δq_i the corresponding displacement in the bar will be

$$\delta q_i \sin \frac{\beta_i x}{l},$$

and the work done by the pulsating load $S \sin \omega t$ during this displacement will be

$$S \sin \omega t \sin \beta_i \delta q_i.$$

Hence the generalized force

$$Q_i = S \sin \omega t \sin \beta_i.$$

Substituting this in solution (107) and performing the integration we obtain

$$(u)_{x=l} = \frac{2gS}{A\gamma l} \sum_{i=1}^{i=\infty} \frac{\sin^2 \beta_i \left(\sin \omega t - \frac{\omega l}{a\beta_i} \sin \frac{a\beta_i t}{l} \right)}{\left(1 + \frac{\sin 2\beta_i}{2\beta_i}\right)\left(\frac{a^2\beta_i^2}{l^2} - \omega^2\right)}. \tag{q}$$

It is seen that the vibration consists of two parts: (1) forced vibrations proportional to *sin* ωt having the same period as the disturbing force and (2) free vibrations proportional to $\sin (a\beta_i t/l)$. When the frequency of the disturbing force approaches one of the natural frequencies of vibration ω approaches the value $a\beta_i/l$ for this mode of vibration and a condition of resonance takes place. The amplitude of vibration of the corresponding term in the series (q) will then increase indefinitely, as was explained before (see pp. 15 and 209). In order to approach the static condition the quantity ω should be considered as small in comparison with $a\beta_i/l$ in the series (q). Neglecting then the terms having $\omega l/a\beta_i$ as a factor, we obtain, for a very slow variation of the pulsating load,

$$(u)_{x=l} = \frac{4lS\sin\omega t}{AE}\sum_{i=1}^{i=\infty}\frac{\sin^2\beta_i}{\beta_i(2\beta_i + \sin 2\beta_i)}, \tag{r}$$

which represents the static elongation of the bar (see eq. 105). By comparing the series (r) and (q) the difference between static and dynamic deflections can be established.* It is seen that a satisfactory record of steam or gas pressure can be obtained only if the frequency of the fundamental mode of vibration of the indicator is high in comparison with the frequency of the pulsating force.

53. Torsional Vibration of Circular Shafts.—*Free Vibration.*—In our previous discussions (see pp. 9 and 253) the mass of the shaft was either neglected or considered small in comparison with the rotating masses attached to the shaft. In the following a more complete theory of the torsional vibrations of a circular shaft with two discs at the ends is given† on the basis of which the accuracy of our previous solution is discussed. It is assumed in the following discussion that the circular cross sections of the shaft during torsional vibration remain plane and the radii of these cross sections remain straight.‡ Let

$GI_p = C$ be torsional rigidity of shaft,
γ be weight per unit of volume of shaft,
θ be angle of twist at any arbitrary cross section mn (see Fig. 175) during torsional vibration,
I_1, I_2 are moments of inertia of the discs at the ends of the shaft about the shaft axis.

Considering an element of the shaft between two adjacent cross sections mn and m_1n_1 the twisting moments at these cross sections will be

$$GI_p\frac{\partial\theta}{\partial x} \qquad \text{and} \qquad GI_p\left(\frac{\partial\theta}{\partial x} + \frac{\partial^2\theta}{\partial x^2}dx\right).$$

The differential equation of rotatory motion of the elemental disc mnm_1n_1 (see Fig. 175) during torsional vibration will be

$$\frac{\gamma I_p}{g}\frac{\partial^2\theta}{\partial t^2} = GI_p\frac{\partial^2\theta}{\partial x^2}$$

* Damping effect is neglected in this consideration.

† See writer's paper in the Bulletin of the Polytechnical Institute in S. Petersburg, 1905, and also his paper "Ueber die Erzwungenen Schwingungen von Prismatischen Stäben," Z. f. Math. u. Phys., Vol. 59 (1911).

‡ A more complete theory can be found in L. Pochhammer's paper, mentioned before (p. 308).

or by using the notation

$$\frac{Gg}{\gamma} = a^2 \tag{109}$$

we obtain

$$\frac{\partial^2 \theta}{\partial t^2} = a^2 \frac{\partial^2 \theta}{\partial x^2}. \tag{110}$$

This equation is identical with the eq. (94) obtained above for the longitudinal vibration and the previous results can be used in various particular cases. For instance, in the case of a shaft with free ends the frequency equation will be identical with eq. (96) and the general solution will be (see eq. 99).

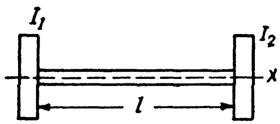

FIG. 178.

$$\theta = \sum_{i=1}^{i=\infty} \cos \frac{i\pi x}{l} \left(A_i \cos \frac{i\pi a t}{l} + B_i \sin \frac{i\pi a t}{l} \right). \tag{111}$$

In the case of a shaft with discs at the ends the problem becomes more complicated and the end conditions must be considered. From the condition that the twisting of the shaft at the ends is produced by the inertia forces of the discs we obtain (see Fig. 178).

$$I_1 \left(\frac{\partial^2 \theta}{\partial t^2} \right)_{x=0} = GI_p \left(\frac{\partial \theta}{\partial x} \right)_{x=0}, \tag{a}$$

$$I_2 \left(\frac{\partial^2 \theta}{\partial t^2} \right)_{x=l} = -GI_p \left(\frac{\partial \theta}{\partial x} \right)_{x=l}. \tag{b}$$

Assume that the shaft performs one of the normal modes of vibration, then it can be written:

$$\theta = X(A \cos pt + B \sin pt), \tag{c}$$

where X is a function of x alone, determining the shape of the mode of vibration under consideration.

Substituting (c) in eq. (110) we obtain

$$a^2 \frac{d^2 X}{dx^2} + p^2 X = 0,$$

from which

$$X = C \cos \frac{px}{a} + D \sin \frac{px}{a}. \tag{d}$$

The constants C and D should be determined in such a manner as to satisfy the end conditions. Substituting (d) in eqs. (a) and (b) we obtain

$$-Cp^2I_1 = D\frac{p}{a}GI_p,$$

$$p^2\left(C\cos\frac{pl}{a} + D\sin\frac{pl}{a}\right)I_2 = \frac{p}{a}GI_p\left(-C\sin\frac{pl}{a} + D\cos\frac{pl}{a}\right). \qquad (e)$$

Eliminating the constants C and D the following *frequency equation* will be obtained,

$$p^2\left(\cos\frac{pl}{a} - \frac{paI_1}{GI_p}\sin\frac{pl}{a}\right)I_2 = -\frac{p}{a}GI_p\left(\sin\frac{pl}{a} + \frac{paI_1}{GI_p}\cos\frac{pl}{a}\right). \qquad (f)$$

Letting

$$\frac{pl}{a} = \beta; \qquad \frac{I_1 g}{\gamma l I_p} = \frac{I_1}{I_0} = m; \qquad \frac{I_2}{I_0} = n, \qquad (g)$$

where $I_0 = (\gamma l I_p/g)$ is the moment of inertia of the shaft about its axis, we obtain, from eq. (f) the frequency equation in the following form:

$$\beta n(1 - m\beta\tan\beta) = -(\tan\beta + m\beta)$$

or

$$\tan\beta = \frac{(m+n)\beta}{mn\beta^2 - 1}. \qquad (112)$$

Let

$$\beta_1, \beta_2, \beta_3, \cdots$$

be the consecutive roots of this transcendental equation, then the corresponding normal functions, from (d) and (e) will be

$$X_i = C_i\left(\cos\frac{\beta_i x}{l} - m\beta_i\sin\frac{\beta_i x}{l}\right)$$

and we obtain for the general solution in this case

$$\theta = \sum_{i=1}^{i=\infty}\left(\cos\frac{\beta_i x}{l} - m\beta_i\sin\frac{\beta_i x}{l}\right)\left(A_i\cos\frac{\beta_i at}{l} + B_i\sin\frac{\beta_i at}{l}\right). \qquad (113)$$

If the moments of inertia I_1 and I_2 of the discs are small in comparison with the moment of inertia I_0 of the shaft, the quantities m and n in eq. 112 become small, the consecutive roots of this equation will approach the values $\pi, 2\pi, \cdots$ and the general solution (113) approaches the solution (111) given above for a shaft with free ends.

Consider now another extreme case, more interesting from a practical standpoint, when I_1 and I_2 are large in comparison with I_0; the quantities m and n will then be large numbers. In this case unity can be neglected in comparison with $mn\beta^2$ in the denominator on the right side of eq. 112 and, instead of eq. (112), we obtain

$$\beta \tan \beta = (1/m + 1/n). \tag{114}$$

This equation is of the same form as eq. (104) (see p. 318) for longitudinal vibrations. The right side of this equation is a small quantity and an approximate solution for the first root will be obtained by substituting $\tan \beta_1 = \beta_1$. Then

$$\beta_1 = \sqrt{1/m + 1/n}. \tag{h}$$

The period of the corresponding mode of vibration, from eq. 113, will be

$$\tau_1 = 2\pi : \frac{\beta_1 a}{l} = \frac{2\pi l}{\beta_1 a}$$

or, by using eqs. 109, (g) and (h), we obtain

$$\tau_1 = 2\pi \sqrt{\frac{lI_1I_2}{GI_p(I_1 + I_2)}}. \tag{115}$$

This result coincides with eq. 16 (see p. 12) obtained by considering the system as having one degree of freedom and neglecting the mass of the shaft.

The approximate values of the consecutive roots of eq. (114) will be,

$$\beta_2 = \pi + 1/\pi(1/m + 1/n); \qquad \beta_3 = 2\pi + 1/2\pi(1/m + 1/n); \cdots.$$

It is seen that all these roots are large in comparison with β_1, and the frequencies of the corresponding modes of vibration will be very high in comparison with the frequency of fundamental type of vibration.

In order to get a closer approximation for the first root of eq. (112), we substitute $\tan \beta_1 = \beta_1 + 1/3\beta_1^3$, then

$$\beta_1 + \frac{1}{3}\beta_1^3 = \frac{\beta_1(m + n)}{mn\beta_1^2 - 1}$$

or

$$\beta_1^2 = \frac{m + n}{\left(mn - \frac{1}{\beta_1^2}\right)\left(1 + \frac{1}{3}\beta_1^2\right)}.$$

Substituting in the right side of this equation the value of β_1 from eq. (*h*) and neglecting small quantities of higher order, we obtain

$$\beta_1 = \sqrt{\frac{1}{m} + \frac{1}{n} - \frac{1}{3}\left(\frac{1}{n^2} + \frac{1}{m^2} - \frac{1}{mn}\right)},$$

and the corresponding frequency of the fundamental vibration will be

$$f_1 = \frac{\beta_1 a}{2\pi l} = \frac{a}{2\pi l}\sqrt{\frac{1}{m} + \frac{1}{n} - \frac{1}{3}\left(\frac{1}{n^2} + \frac{1}{m^2} - \frac{1}{mn}\right)}. \tag{116}$$

The same result will be obtained if in the first approximation for the frequency

$$f = \frac{1}{2\pi}\sqrt{\frac{GI_p(I_1 + I_2)}{lI_1I_2}}$$

as obtained from eq. (115) we substitute

$$I_1 + \frac{1}{3} I_0 \frac{I_2}{I_1 + I_2} \quad \text{and} \quad I_2 + \frac{1}{3} I_0 \frac{I_1}{I_1 + I_2} \text{for } I_1 \text{ and } I_2.$$

This means that the second approximation (116) coincides with the result which would have been obtained by the Rayleigh method (see Art. 16, p. 88). According to this method one third of the moment of inertia of the part of the shaft between the disc and the nodal cross section should be added to the moment of inertia of each disc. This approximation is always sufficient in practical applications for calculating the frequency of the fundamental mode of vibration.*

Forced Vibration.—In studying forced torsional vibrations generalized coordinates again are very useful. Considering the brackets containing t in the general solution (113) as such coordinates, we obtain

$$\theta = \sum_{i=1}^{i=\infty} q_i\left(\cos\frac{\beta_i x}{l} - m\beta_i \sin\frac{\beta_i x}{l}\right), \tag{117}$$

in which β_i are consecutive roots of eq. (112).

* A graphical method for determining the natural frequencies of torsional vibration of shafts with discs has been developed by F. M. Lewis, see papers: "Torsional Vibrations of Irregular Shafts," Journal Am. Soc. of Naval Engs. Nov. 1919, p. 857 and "Critical Speeds of Torsional Vibration," Journal Soc. Automotive Engs., Nov. 1920, p. 413.

The potential energy of the system will be

$$V = \frac{GI_p}{2}\int_0^l \left(\frac{\partial\theta}{\partial x}\right)^2 dx = \frac{GI_p}{2}\int_0^l \left\{\sum_{i=1}^{i=\infty}\frac{\beta_i}{l}\, q_i\left(\sin\frac{\beta_i x}{l} + m\beta_i\cos\frac{\beta_i x}{l}\right)\right\}^2 dx$$

$$= \frac{GI_p}{8l}\sum_{i=1}^{i=\infty} A_i\beta_i q_i^2, \tag{118}$$

where

$$A_i = 2\beta_i(1 + m^2\beta_i^2) - \sin 2\beta_i + m^2\beta_i^2 \sin 2\beta_i + 2\beta_i m(1 - \cos 2\beta_i). \qquad (k)$$

The terms containing products of the coordinates in expression (118) disappear in the process of integration in virtue of eq. (112). Such a result should be expected if we remember that our generalized coordinates are principal or normal coordinates of the system.

The kinetic energy of the system consists of the energy of the vibrating shaft and of the energies of the two oscillating discs:

$$T = \frac{\gamma I_p}{2g}\int_0^l \dot{\theta}^2 dx + \frac{1}{2}I_1\dot{\theta}^2_{x=0} + \frac{1}{2}I_2\dot{\theta}^2_{x=l}$$

or, substituting (117) for θ we obtain

$$T = \frac{I_0}{8}\sum_{i=1}^{i=\infty} A_i\frac{1}{\beta_i}\dot{q}_i^2, \tag{119}$$

in which A_i is given by eq. (k).

By using eqs. (118) and (119) Lagrange's equations will become:

$$\frac{I_0}{4\beta_i}\ddot{q}_i + \frac{GI_p\beta_i}{4l}q_i = \frac{1}{A_i}Q_i$$

or

$$\ddot{q}_i + \frac{a^2\beta_i^2}{l^2}q_i = \frac{4\beta_i}{I_0A_i}Q_i, \qquad (l)$$

in which Q_i is the symbol for the generalized force corresponding to the generalized coordinate q_i.

Considering only the vibration produced by the disturbing force, we obtain from eq. (l)

$$q_i = \frac{4l}{aI_0A_i}\int_0^t Q_i \sin\frac{a\beta_i}{l}(t - t_1)dt_1.$$

Substituting in eq. (117), the general expression for the vibrations produced by the disturbing forces, we will find:

$$\theta=\frac{4l}{aI_0}\sum_{i=1}^{i=\infty}\frac{1}{A_i}\left(\cos\frac{\beta_i x}{l}-m\beta_i\sin\frac{\beta_i x}{l}\right)\int_0^t Q_i\sin\frac{a\beta_i}{l}(t-t_1)dt_1. \qquad (120)$$

In every particular case it remains only to substitute for Q_i the corresponding expression and to perform the indicated integration in order to obtain forced vibrations. These forced vibrations have the tendency to increase indefinitely* when the period of the disturbing force coincides with the period of one of the natural vibrations.

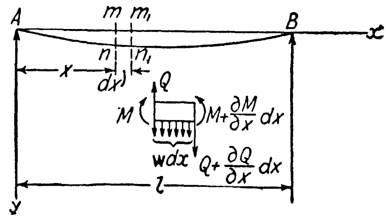

FIG. 179.

54. Lateral Vibration of Prismatical Bars.—*Differential Equation of Lateral Vibration.*—Assuming that vibration occurs in one of the principal planes of flexure of the bar and that cross sectional dimensions are small in comparison with the length of the bar, the well known differential equation of the deflection curve

$$EI\frac{d^2y}{dx^2}=-M \qquad (121)$$

will now be used, in which

EI is flexural rigidity and,

M is bending moment at any cross section. The direction of the axes and the positive directions of bending moments and shearing forces are as shown in Fig. 179.

Differentiating eq. (121) twice we obtain

$$\begin{aligned}\frac{d}{dx}\left(EI\frac{d^2y}{dx^2}\right)&=-\frac{dM}{dx}=-Q,\\ \frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right)&=-\frac{dQ}{dx}=w.\end{aligned} \qquad (a)$$

This last equation representing the differential equation of a bar subjected to a continuous load of intensity w can be used also for obtaining the equation of lateral vibration. It is only necessary to apply D'Alembert's

* Damping is neglected in our calculations.

principle and to imagine that the vibrating bar is loaded by inertia forces, the intensity of which varies along the length of the bar and is given by

$$-\frac{\gamma A}{g}\frac{\partial^2 y}{\partial t^2}, \tag{b}$$

where γ is the weight of material of the bar per unit volume, and A is cross-sectional area.

Substituting (*b*) for w in eq. (*a*) the general equation for the lateral vibration of the bar becomes *

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right) = -\frac{\gamma A}{g}\frac{\partial^2 y}{\partial t^2}. \tag{122}$$

In the particular case of a prismatical bar the flexural rigidity EI remains constant along the length of the bar and we obtain from eq. (122)

$$EI\frac{\partial^4 y}{\partial x^4} = -\frac{\gamma A}{g}\frac{\partial^2 y}{\partial t^2}$$

or

$$\frac{\partial^2 y}{\partial t^2} + a^2\frac{\partial^4 y}{\partial x^4} = 0, \tag{123}$$

in which

$$a^2 = \frac{EIg}{A\gamma}. \tag{124}$$

We begin with studying the *normal modes* of vibration. When a bar performs a normal mode of vibration the deflection at any location varies harmonically with the time and can be represented as follows:

$$y = X(A\cos pt + B\sin pt), \tag{c}$$

where X is a function of the coordinate x determining the shape of the normal mode of vibration under consideration. Such functions are called "normal functions." Substituting (*c*) in eq. (123), we obtain,

$$\frac{d^4X}{dx^4} = \frac{p^2}{a^2}X, \tag{125}$$

* The differential equation in which damping is taken into consideration has been discussed by H. Holzer, Zeitschr. f. angew. Math. u. Mech., V. 8, p. 272, 1928. See also K. Sezawa, Zeitschr f. angew. Math. u. Mech., V. 12, p. 275, 1932.

from which the normal functions for any particular case can be obtained.

By using the notation

$$\frac{p^2}{a^2} = \frac{p^2 A\gamma}{EIg} = k^4 \tag{126}$$

it can be easily verified that $\sin kx$, $\cos kx$, $\sinh kx$ and $\cosh kx$ will be particular solutions of eq. (125) and the general solution of this equation will be obtained in the form,

$$X = C_1 \sin kx + C_2 \cos kx + C_3 \sinh kx + C_4 \cosh kx, \tag{127}$$

in which $C_1, \cdots C_4$ are constants which should be determined in every particular case from the conditions at the ends of the bar. At an end which is simply supported, i.e., where the deflection and bending moment are equal to zero, we have

$$X = 0; \qquad \frac{d^2X}{dx^2} = 0. \tag{d}$$

At a built-in end, i.e., where the deflection and slope of the deflection curve are equal to zero,

$$X = 0; \qquad \frac{dX}{dx} = 0. \tag{e}$$

At a free end the bending moment and the shearing force both are equal to zero and we obtain,

$$\frac{d^2X}{dx^2} = 0; \qquad \frac{d^3X}{dx^3} = 0. \tag{f}$$

For the two ends of a vibrating bar we always will have four end conditions from which the ratios between the arbitrary constants of the general solution (127) and the *frequency equation* can be obtained. In this manner the modes of natural vibration and their frequencies will be established. By superimposing all possible normal vibrations (*c*) the general expression for the free lateral vibrations becomes:

$$y = \sum_{i=1}^{i=\infty} X_i(A_i \cos p_i t + B_i \sin p_i t) \cdots. \tag{128}$$

Applications of this general theory to particular cases will be considered later.

Forced Vibration.—In considering forced lateral vibrations of bars *generalized coordinates* are very useful and, in the following, the expressions in the brackets of eq. (128) will be taken as such coordinates. Denoting them by the symbol, q_i we obtain

$$y = \sum_{i=1}^{i=\infty} q_i X_i. \tag{129}$$

In order to derive Lagrange's equations it is necessary to find expressions for the potential and kinetic energy.

The potential energy of the system is the energy of bending and can be calculated as follows:

$$V = \frac{EI}{2} \int_0^l \left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx = \frac{EI}{2} \sum_{i=1}^{i=\infty} q_i^2 \int_0^l \left(\frac{d^2 X_i}{dx^2}\right)^2 dx. \tag{130}$$

The kinetic energy of the vibrating bar will be

$$T = \frac{\gamma A}{2g} \int_0^l \dot{y}^2 dx = \frac{\gamma A}{2g} \sum_{i=1}^{i=\infty} \dot{q}_i^2 \int_0^l X_i^2 dx. \tag{131}$$

The terms containing products of the coordinates disappear from the expressions (130) and (131) in virtue of the fundamental property of normal functions (see p. 209). This can also be proven by direct integration.

Let X_m and X_n be two normal functions corresponding to normal modes of vibration of the order m and n, having frequencies $p_m/2\pi$ and $p_n/2\pi$. Substituting in eq. (125) we obtain

$$\frac{d^4 X_m}{dx^4} = \frac{p_m^2}{a^2} X_m,$$

$$\frac{d^4 X_n}{dx^4} = \frac{p_n^2}{a^2} X_n.$$

Multiplying the first of these equations with X_n and the second with X_m, subtracting one from another and integrating we have

$$\frac{p^2{}_n - p^2{}_m}{a^2} \int_0^l X_m X_n dx = \int_0^l \left(X_m \frac{d^4 X_n}{dx^4} - X_n \frac{d^4 X_m}{dx^4}\right) dx,$$

from which, by integration by parts, follows

$$\frac{p_n^2 - p_m^2}{a^2} \int_0^l X_m X_n dx = \left| X_m \frac{d^3 X_n}{dx^3} - X_n \frac{d^3 X_m}{dx^3} \right.$$

$$\left. + \frac{dX_n}{dx}\frac{d^2 X_m}{dx^2} - \frac{dX_m}{dx}\frac{d^2 X_n}{dx^2} \right|_0^l \cdot \cdot \cdot. \tag{132}$$

From the end conditions (*d*), (*e*) and (*f*) it can be concluded that in all cases the right side of the above equation is equal to zero, hence,

$$\int_0^l X_m X_n dx = 0 \text{ when } m \neq n$$

and the terms containing the products of the coordinates disappear from eq. (131). By using an analogous method is can be shown also that the products of the coordinates disappear from eq. (130).

Equation (132) can be used also for the calculation of integrals such as

$$\int_0^l X_m{}^2 dx \quad \text{and} \quad \int_0^l \left(\frac{d^2X_m}{dx^2}\right)^2 dx \tag{g}$$

entering into the expressions (130) and (131) of the potential and the kinetic energy of a vibrating bar.

It is easy to see that by directly substituting $m = n$ into this equation, the necessary results cannot be obtained because both sides of the equation become equal to zero. Therefore the following procedure should be adopted for calculating the integrals (*g*). Substitute for X_n in eq. (132) a function which is very near to the function X_m and which will be obtained from eqs. (125) and (126) by giving to the quantity k an infinitely small increase δk, so that X_n approaches X_m when δk approaches zero. Then

$$\frac{p_n{}^2}{a^2} = (k + \delta k)^4 = k^4 + 4k^3\delta k,$$

$$\frac{p_n{}^2 - p_m{}^2}{a^2} = 4k^3\delta k,$$

$$X_n = X_m + \frac{dX_m}{dk}\delta k.$$

Substituting in eq. (132) and neglecting small quantities of higher order we obtain

$$4k^3 \int_0^l X_m{}^2 dx = \left| X_m \frac{d}{dk}\frac{d^3X_m}{dx^3} - \frac{dX_m}{dk}\frac{d^3X_m}{dx^3} + \frac{d}{dk}\left(\frac{dX_m}{dx}\right)\frac{d^2X_m}{dx^2} - \frac{dX_m}{dx}\frac{d}{dk}\left(\frac{d^2X_m}{dx^2}\right)\right|_0^l \tag{h}$$

In the following we denote by X', X'', $\cdots$ consecutive derivatives of X with respect to kx, then

$$\frac{dX_m}{dx} = kX_m{}'; \quad \frac{dX_m}{dk} = xX_m{}'.$$

With these notations eq. (125) becomes

$$X'''' = X,$$

and eq. (*h*) will have the following form:

$$4k^3\int_0^l X^2_m dx = \Big|3X_m k^2 X_m''' + k^3 x X^2_m - k^3 x X_m' X_m''' +$$
$$+ k^2 X_m''(X_m' + kxX_m'') - kX_m'(2kX_m'' + k^2 x X_m''')\Big|_0^l$$

or

$$4k\int_0^l X^2_m dx = \Big|3X_m X_m''' + kxX^2_m - 2kxX_m'X_m''' - X_m'X_m'' + kx(X_m'')^2\Big|_0^l. \qquad (k)$$

From the end conditions (d), (e) and (f) it is easy to see that the terms in eq. (k) containing the products $X_m X_m'''$ and $X_m'X_m''$ are equal to zero for any manner of fastening the ends, hence

$$\int_0^l X^2_m dx = \tfrac{1}{4}\Big|x\{X^2_m - 2X_m'X_m''' + (X_m'')^2\}\Big|_0^l$$
$$= \frac{l}{4}\{X^2_m - 2X_m'X_m''' + (X_m'')^2\}_{x=l}. \qquad (133)$$

From this equation the first of the integrals (g) easily can be calculated for any kind of fastening of the ends of the bar. If the right end ($x = l$) of the bar is free,

$$(X_m'')_{x=l} = 0; \quad (X_m''')_{x=l} = 0,$$

and we obtain, from (133)

$$\int_0^l X^2_m dx = \frac{l}{4}(X_m^2)_{x=l}. \qquad (134)$$

If the same end is built in, we obtain

$$\int_0^l X^2_m dx = \frac{l}{4}(X_m'')^2_{x=l}. \qquad (135)$$

For the hinged end we obtain

$$\int_0^l X^2_m dx = -\frac{l}{2}(X_m'X_m''')_{x=l}. \qquad (136)$$

In calculating the second of the integrals (g) equation (125) should be used. Multiplying this equation by X and integrating along the length of the bar:

$$\frac{p^2}{a^2}\int_0^l X^2 dx = \int_0^l \frac{d^4X}{dx^4}Xdx.$$

Integrating the right side of this equation by parts we obtain,

$$\int_0^l \left(\frac{d^2X}{dx^2}\right)^2 dx = \frac{p^2}{a^2}\int_0^l X^2 dx. \qquad (137)$$

This result together with eq. (133) gives us the second of the integrals (g) and now the expressions (130) and (131) for V and T can be calculated. Eqs. (133) and (137) are very useful in investigating forced vibrations of bars with other end conditions than hinged ones.

55. The Effect of Shearing Force and Rotatory Inertia.—In the previous discussion the cross sectional dimensions of the bar were considered to be very small in comparison with the length and the simple equation (121) was used for the deflection curve. Corrections will now be given, taking into account the effect of the cross sectional dimensions on the frequency. These corrections may be of considerable importance in studying the modes of vibration of higher frequencies when a vibrating bar is subdivided by *nodal cross sections* into comparatively short portions.

*Rotatory Inertia.**—It is easy to see that during vibration the elements of the bar such as mnm_1n_1 (see Fig. 179) perform not only a translatory motion but also rotate. The variable angle of rotation which is equal to the slope of the deflection curve will be expressed by $\partial y/\partial x$ and the corresponding angular velocity and angular acceleration will be given by

$$\frac{\partial^2 y}{\partial x \partial t} \quad \text{and} \quad \frac{d^3 y}{\partial x \partial t^2}.$$

Therefore the moment of the inertia forces of the element mnm_1n_1 about the axis through its center of gravity and perpendicular to the xy plane will be †

$$-\frac{I\gamma}{g}\frac{\partial^3 y}{\partial x \partial t^2}dx.$$

This moment should be taken into account in considering the variation in bending moment along the axis of the bar. Then, instead of the first of the equations (*a*) p. 331, we will have,

$$\frac{dM}{dx} = Q - \frac{I\gamma}{g}\frac{\partial^3 y}{\partial x \partial t^2}. \tag{a}$$

Substituting this value of dM/dx in the equation for the deflection curve

$$EI\frac{d^4 y}{dx^4} = -\frac{d^2 M}{dx^2},$$

and using (*b*) p. 332, we obtain

$$EI\frac{\partial^4 y}{\partial x^4} = -\frac{\gamma A}{g}\frac{\partial^2 y}{\partial t^2} + \frac{I\gamma}{g}\frac{\partial^4 y}{\partial x^2 \partial t^2}. \tag{138}$$

This is the differential equation for the lateral vibration of prismatical bars in which the second term on the right side represents the effect of rotatory inertia.

Effect of Shearing Force.‡—A still more accurate differential equation is obtained if not only the rotatory inertia, but also the deflection due to shear will be taken into account. The slope of the deflection curve depends not only on the rotation of cross sections of the bar but also on the shear. Let ψ denote the slope of the deflection curve when the shearing force is neglected and β the angle of shear at the neutral axis in the same cross section, then we find for the total slope

$$\frac{dy}{dx} = \psi + \beta.$$

* See Lord Rayleigh, "Theory of Sound," paragraph 186.

† The moment is taken positive when it is a clockwise direction.

‡ See writer's paper in Philosophical Magazine (Ser. 6) Vol. 41, p. 744 and Vol. 43, p. 125.

From the elementary theory of bending we have for bending moment and shearing force the following equations,

$$M = -EI\frac{d\psi}{dx}; \quad Q = k'\beta AG = k'\left(\frac{dy}{dx} - \psi\right)AG, \tag{b}$$

in which k' is a numerical factor depending on the shape of the cross section; A is the cross sectional area and G is modulus of elasticity in shear. The differential equation of rotation of an element mnm_1n_1 (Fig. 179) will be

$$-\frac{\partial M}{\partial x}dx + Qdx = \frac{I\gamma}{g}\frac{\partial^2\psi}{\partial t^2}dx.$$

Substituting (*b*) we obtain

$$EI\frac{\partial^2\psi}{\partial x^2} + k'\left(\frac{\partial y}{\partial x} - \psi\right)AG - \frac{I\gamma}{g}\frac{\partial^2\psi}{\partial t^2} = 0. \tag{c}$$

The differential equation for the translatory motion of the same element in a vertical direction will be

$$\frac{\partial Q}{\partial x}dx = \frac{\gamma A}{g}\frac{\partial^2 y}{\partial t^2}dx,$$

or

$$\frac{\gamma A}{g}\frac{\partial^2 y}{\partial t^2} - k'\left(\frac{\partial^2 y}{\partial x^2} - \frac{\partial\psi}{\partial x}\right)AG = 0. \tag{d}$$

Eliminating ψ from equations (*c*) and (*d*) the following more complete differential equation for the lateral vibration of prismatical bars will be obtained

$$EI\frac{\partial^4 y}{\partial x^4} + \frac{\gamma A}{g}\frac{\partial^2 y}{\partial t^2} - \left(\frac{\gamma I}{g} + \frac{EI\gamma}{gk'G}\right)\frac{\partial^4 y}{\partial x^2\partial t^2} + \frac{\gamma I}{g}\frac{\gamma}{gk'G}\frac{\partial^4 y}{\partial t^4} = 0. \tag{139}$$

The application of this equation in calculating the frequencies will be shown in the following article.

56. Free Vibration of a Bar with Hinged Ends.—*General Solution.*—In considering particular cases of vibration it is useful to present the general solution (127) in the following form

$$X = C_1(\cos kx + \cosh kx) + C_2(\cos kx - \cosh kx) + C_3(\sin kx + \sinh kx) + C_4(\sin kx - \sinh kx) \cdots. \tag{140}$$

In the case of hinged ends the end conditions are

$$(1)\quad (X)_{x=0} = 0; \qquad (2)\quad \left(\frac{d^2X}{dx^2}\right)_{x=0} = 0;$$

$$(3)\quad (X)_{x=l} = 0; \qquad (4)\quad \left(\frac{d^2X}{dx^2}\right)_{x=l} = 0. \tag{a}$$

From the first two conditions (*a*) it can be concluded that the constants C_1 and C_2 in solution (140) should be taken equal to zero. From conditions (3) and (4) we obtain $C_3 = C_4$ and

$$\sin kl = 0, \tag{141}$$

which is the *frequency equation* for the case under consideration. The consecutive roots of this equation are

$$kl = \pi,\ 2\pi,\ 3\pi\ \cdots. \tag{142}$$

The circular frequencies of the consecutive modes of vibration will be obtained from eq. (126):

$$p_1 = ak_1^2 = \frac{a\pi^2}{l^2}; \qquad p_2 = \frac{4a\pi^2}{l^2}; \qquad p_3 = \frac{9a\pi^2}{l^2}; \cdots, \tag{143}$$

and the frequency f_n of any mode of vibration will be found from the equation

$$f_n = \frac{p_n}{2\pi} = \frac{n^2 a\pi}{2l^2} = \frac{\pi n^2}{2l^2}\sqrt{\frac{EIg}{A\gamma}}. \tag{144}$$

The corresponding period of vibration will be

$$\tau_n = \frac{1}{f_n} = \frac{2l^2}{\pi n^2}\sqrt{\frac{A\gamma}{EIg}}. \tag{145}$$

It is seen that the period of vibration is proportional to the square of the length and inversely proportional to the radius of gyration of the cross section. For geometrically similar bars the periods of vibration increase in the same proportion as the linear dimensions.

In the case of rotating circular shafts of uniform cross section the frequencies calculated by eq. 144 represent the critical numbers of revolutions per second. When the speed of rotation of the shaft approaches one of the frequencies (144) a considerable lateral vibration of the shaft should be expected.

The shape of the deflection curve for the various modes of vibration is determined by the normal function (140). It was shown that in the case we are considering, $C_1 = C_2 = 0$ and $C_3 = C_4$, hence the normal function has a form

$$X = D \sin kx. \tag{b}$$

Substituting for k its values, from eq. (142), we obtain

$$X_1 = D_1 \sin\frac{\pi x}{l}; \qquad X_2 = D_2 \sin\frac{2\pi x}{l}; \qquad X_3 = D_3 \sin\frac{3\pi x}{l}; \cdots.$$

It is seen that the deflection curve during vibration is a sine curve, the number of half waves in the consecutive modes of vibration being equal to 1, 2, 3 $\cdots$. By superimposing such sinusoidal vibrations any kind of free vibration due to any initial conditions can be represented. Substituting (b) in the general solution (128) we obtain

$$y = \sum_{i=1}^{i=\infty} \sin \frac{i\pi x}{l} (C_i \cos p_i t + D_i \sin p_i t). \tag{146}$$

The constants C_i, D_i, of this solution should be determined in every particular case so as to satisfy the initial conditions. Assume, for instance, that the initial deflections and initial velocities along the bar are given by the equations

$$(y)_{t=0} = f(x) \qquad \text{and } (\dot{y})_{t=0} = f_1(x).$$

Substituting $t = 0$ in expression (146) and in the derivative of this expression with respect to t, we obtain,

$$(y)_{t=0} = f(x) = \sum_{i=1}^{i=\infty} C_i \sin \frac{i\pi x}{l}, \tag{c}$$

$$(\dot{y})_{t=0} = f_1(x) = \sum_{i=1}^{i=\infty} p_i D_i \sin \frac{i\pi x}{l}. \tag{d}$$

Now the constants C_i and D_i can be calculated in the usual way by multiplying (c) and (d) by $\sin (i\pi x/l)dx$ and by integrating both sides of these equations from $x = 0$ to $x = l$. In this manner we obtain

$$C_i = \frac{2}{l} \int_0^l f(x) \sin \frac{i\pi x}{l} dx, \tag{e}$$

$$D_i = \frac{2}{lp_i} \int_0^l f_1(x) \sin \frac{i\pi x}{l} dx. \tag{f}$$

Assume, for instance, that in the initial moment the axis of the bar is straight and that due to impact an initial velocity v is given to a short portion δ of the bar at the distance c from the left support. Then, $f(x) = 0$ and $f_1(x)$ also is equal to zero in all points except the point $x = c$ for which $f_1(c) = v$. Substituting this in the eqs. (e) and (f) we obtain,

$$C_i = 0; \qquad D_i = \frac{2}{lp_i} v\delta \sin \frac{i\pi c}{l}.$$

Substituting in (146)

$$y = \frac{2v\delta}{l} \sum_{i=1}^{i=\infty} \frac{1}{p_i} \sin \frac{i\pi c}{l} \sin \frac{i\pi x}{l} \sin p_i t. \tag{147}$$

If $c = (l/2)$, i.e., the impact is produced at the middle of the span,

$$\begin{aligned} y &= \frac{2v\delta}{l}\left(\frac{1}{p_1}\sin\frac{\pi x}{l}\sin p_1 t - \frac{1}{p_3}\sin\frac{3\pi x}{l}\sin p_3 t + \frac{1}{p_5}\sin\frac{5\pi x}{l}\sin p_5 t - \cdots\right) \\ &= \frac{2v\delta l}{a\pi^2}\left(\frac{1}{1}\sin\frac{\pi x}{l}\sin p_1 t - \frac{1}{9}\sin\frac{3\pi x}{l}\sin p_3 t + \frac{1}{25}\sin\frac{5\pi x}{l}\sin p_5 t - \cdots\right) \end{aligned} \tag{g}$$

It is seen that in this case only modes of vibration symmetrical about the middle of the span will be produced and the amplitudes of consecutive modes of vibration entering in eq. (g) decrease as $1/i^2$.

The Effect of Rotatory Inertia and of Shear.—In order to find the values of the frequencies more accurately equation (139) instead of equation (123) should be taken. Dividing eq. (139) by $A\gamma/g$ and using the notation,

$$r^2 = \frac{I}{A}, \tag{h}$$

we obtain

$$a^2\frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial t^2} - r^2\left(1 + \frac{E}{k'G}\right)\frac{\partial^4 y}{\partial x^2 \partial t^2} + r^2\frac{\gamma}{gk'G}\frac{\partial^4 y}{\partial t^4} = 0. \tag{148}$$

This equation and the end conditions will be satisfied by taking

$$y = C\sin\frac{m\pi x}{l}\cos p_m t. \tag{k}$$

Substituting in eq. (148) we obtain the following equation for calculating the frequencies

$$a^2\frac{m^4\pi^4}{l^4} - p_m{}^2 - p_m{}^2\frac{m^2\pi^2 r^2}{l^2} - p_m{}^2\frac{m^2\pi^2 r^2}{l^2}\frac{E}{k'G} + \frac{r^2\gamma}{gk'G}p_m{}^4 = 0. \tag{149}$$

Considering only the first two terms in this equation we have

$$p_m = a\frac{m^2\pi^2}{l^2} = \frac{a\pi^2}{\lambda^2}, \tag{l}$$

in which

$\lambda = (l/m)$ is the length of the half waves in which the bar is subdivided during vibration.

This coincides with the result (143) obtained before. By taking the three first terms in eq. (149) and considering $\pi^2 r^2/\lambda^2$ as a small quantity we obtain

$$p_m = \frac{a\pi^2}{\lambda^2}\left(1 - \frac{\pi^2 r^2}{2\lambda^2}\right). \tag{m}$$

In this manner the *effect of rotatory inertia* is taken into account and we see that this correction becomes more and more important with a decrease of λ, i.e., with an increase in the frequency of vibration.

In order to obtain the *effect of shear* all terms of eq. (149) should be taken into consideration. Substituting the first approximation (l) for p_m in the last term of this equation it can be shown that this term is a small quantity of the second order as compared with the small quantity π^2r^2/λ^2. Neglecting this term we obtain,*

$$p_m = \frac{a\pi^2}{\lambda^2}\left\{1 - \frac{1}{2}\frac{\pi^2r^2}{\lambda^2}\left(1 + \frac{E}{k'G}\right)\right\}. \tag{150}$$

Assuming $E = 8/3G$ and taking a bar of rectangular cross section for which $k' = 2/3$, we have

$$\frac{E}{k'G} = 4.$$

The correction due to shear is four times larger than the correction due to rotatory inertia.

Assuming that the wave length λ is ten times larger than the depth of the beam, we obtain

$$\frac{1}{2}\cdot\frac{\pi^2r^2}{\lambda^2} = \frac{1}{2}\cdot\frac{\pi^2}{12}\cdot\frac{1}{100} = .004,$$

and the correction for rotatory inertia and shear together will be about 2 per cent.

57. Other End Fastenings.—*Bar with Free Ends.*—In this case we have the following end conditions:

$$(1)\quad \left(\frac{d^2X}{dx^2}\right)_{x=0} = 0; \qquad (2)\quad \left(\frac{d^3X}{dx^3}\right)_{x=0} = 0;$$

$$(3)\quad \left(\frac{d^2X}{dx^2}\right)_{x=l} = 0; \qquad (4)\quad \left(\frac{d^3X}{dx^3}\right)_{x=l} = 0. \tag{a}$$

In order to satisfy the conditions (1) and (2) we have to take in the general solution (140)

$$C_2 = C_4 = 0$$

so that

$$X = C_1(\cos kx + \cosh kx) + C_3(\sin kx + \sinh kx). \tag{b}$$

From the conditions (3) and (4) we obtain

$$\begin{aligned} C_1(-\cos kl + \cosh kl) + C_3(-\sin kl + \sinh kl) &= 0,\\ C_1(\sin kl + \sinh kl) + C_3(-\cos kl + \cosh kl) &= 0. \end{aligned} \tag{c}$$

A solution for the constants C_1 and C_3, different from zero, can be obtained only in the case when the determinant of equations (*c*) is equal to zero. In this manner the following *frequency equation* is obtained:

$$(-\cos kl + \cosh kl)^2 - (\sinh^2 kl - \sin^2 kl) = 0$$

* This result is in a very satisfactory agreement with experiments. See paper by E. Goens, Annalen der Physik 5 series, Vol. 11, p. 649, 1931.

or, remembering that

$$\cosh^2 kl - \sinh^2 kl = 1,$$
$$\cos^2 kl + \sin^2 kl = 1,$$

we have

$$\cos kl \cosh kl = 1. \tag{151}$$

The first six consecutive roots of this equation are given in the table below:

k_1l	k_2l	k_3l	k_4l	k_5l	k_6l
0	4.730	7.853	10.996	14.137	17.279

Now the frequencies can be calculated by using eq. (126)

$$f_1 = 0; \qquad f_2 = \frac{p_2}{2\pi} = \frac{k_2{}^2 a}{2\pi}; \qquad f_3 = \frac{p_3}{2\pi} = \frac{k_3{}^2 a}{2\pi}, \cdots.$$

Substituting the consecutive roots of eq. (151) in eq. (*c*) the ratios C_1/C_3 for the corresponding modes of vibration can be calculated and the shape of the deflection curve during vibration will then be obtained from eq. (*b*). In the Fig. 180 below the first three modes of natural vibration are shown. On these vibrations a displacement of the bar as a rigid body can be superposed. This displacement corresponds to the frequency $k_1l = 0$. Then the right side of eq. (125) becomes zero and by taking into consideration the end conditions (*a*), we obtain $X = a + bx$. The corresponding motion can be investigated in the same manner as was shown in the case of longitudinal vibration (see p. 316).

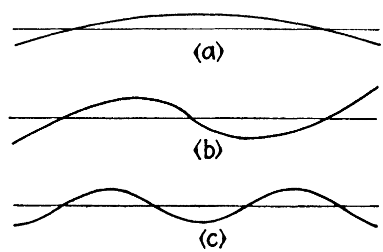

FIG. 180.

Bar with Built-in Ends.—The end conditions in this case are:

$$(1)\quad (X)_{x=0} = 0; \qquad (2)\quad \left(\frac{dX}{dx}\right)_{x=0} = 0;$$

$$(3)\quad (X)_{x=l} = 0; \qquad (4)\quad \left(\frac{dX}{dx}\right)_{x=l} = 0. \tag{d}$$

The first two conditions will be satisfied if in the general solution (140) we take

$$C_1 = C_3 = 0.$$

From the two other conditions the following equations will be obtained

$$\begin{aligned} C_2(\cos kl - \cosh kl) + C_4(\sin kl - \sinh kl) &= 0, \\ C_2(\sin kl + \sinh kl) + C_4(-\cos kl + \cosh kl) &= 0, \end{aligned}$$

from which the same frequency equation as above (see eq. (151)) can be deduced. This means that the consecutive frequencies of vibration of a bar with built-in ends are the same as for the same bar with free ends.*

Bar with One End Built in, Other End Free.—Assuming that the left end ($x = 0$) is built in, the following end conditions will be obtained:

$$(1) \quad (X)_{x=0} = 0; \qquad (2) \quad \left(\frac{dX}{dx}\right)_{x=0} = 0;$$

$$(3) \quad \left(\frac{d^2X}{dx^2}\right)_{x=l} = 0; \qquad (4) \quad \left(\frac{d^3X}{dx^3}\right)_{x=l} = 0.$$

From the first two conditions we conclude that $C_1 = C_3 = 0$ in the general solution (140). The remaining two conditions give us the following frequency equation:

$$\cos kl \cosh kl = -1.$$

The consecutive roots of this equation are given in the table below:

k_1l	k_2l	k_3l	k_4l	k_5l	k_6l
1.875	4.694	7.855	10.996	14.137	17.279

It is seen that with increasing frequency these roots approach the roots obtained above for a bar with free ends. The frequency of vibration of any mode will be

$$f_i = \frac{p_i}{2\pi} = \frac{ak_i^2}{2\pi}.$$

Taking, for instance, the fundamental mode of vibration, we obtain

$$f_1 = \frac{a}{2\pi}\left(\frac{1.875}{l}\right)^2.$$

The corresponding period of vibration will be:

$$\tau_1 = \frac{1}{f_1} = \frac{2\pi}{a}\frac{l^2}{1.875^2} = \frac{2\pi}{3.515}\sqrt{\frac{A\gamma l^4}{EIg}}.$$

* From eq. (125), it can be concluded that in this case there is no motion corresponding to $k_i l = 0$.

This differs by less than 1.5 per cent from the approximate solution obtained by using Rayleigh's method (see p. 86).

Bar with One End Built in, Other End Supported.—In this case the frequency equation will be

$$\tan kl = \tanh kl.$$

The consecutive roots of this equation are:

k_1l	k_2l	k_3l	k_4l	k_5l
3.927	7.069	10.210	13.352	16.493

*Beam on Many Supports.**—Consider the case of a continuous beam with n spans simply supported at the ends and at $(n-1)$ intermediate supports. Let $l_1, l_2, \cdots, l_n$ the lengths of consecutive spans, the flexural rigidity of the beam being the same for all spans. Taking the origin of coordinates at the left end of each span, solution (127) p. 333 will be used for the shape of the deflection curve of each span during vibration. Taking into consideration that the deflection at the left end $(x = 0)$ is equal to zero the normal function for the span r will be

$$X_r = a_r(\cos kx - \cosh kx) + c_r \sin kx + d_r \sinh kx, \tag{e}$$

in which a_r, c_r and d_r are arbitrary constants. The consecutive derivatives of (e) will be

$$X_r' = -a_rk(\sin kx + \sinh kx) + c_rk \cos kx + d_rk \cosh kx, \tag{f}$$

$$X_r'' = -a_rk^2(\cos kx + \cosh kx) - c_rk^2 \sin kx + d_rk^2 \sinh kx. \tag{g}$$

Substituting $x = 0$ in eqs. (f) and (g) we obtain

$$(X_r')_{x=0} = k(c_r + d_r); \qquad (X_r'')_{x=0} = -2k^2a_r.$$

It is seen that $c_r + d_r$ is proportional to the slope of the deflection curve, and a_r is proportional to the bending moment at the support r. From the conditions of simply supported ends it can now be concluded that $a_1 = a_{n+1} = 0$.

Considering the conditions at the right end of the span r we have,

$$(X_r)_{x=l_r} = 0; \quad (X'_r)_{x=l_r} = (X'_{r+1})_{x=0}; \quad (X_r'')_{x=l_r} = (X''_{r+1})_{x=0},$$

* See E. R. Darnley, Phil. Mag., Vol. 41 (1921), p. 81. See also D. M. Smith, Engineering, Vol. 120 (1925), p. 808; K. Hohenemser and W. Prager, "Dynamik der Stabwerke," p. 127. Berlin 1933; K. Federhofer, Bautechn., Vol. 11, p. 647, 1933; F. Stüssi, Schweiz. Bauztg., Vol. 104, p. 189, 1934, and W. Mudrak, Ingenieur-Archiv, Vol. 7, p. 51, 1936.

or by using (e), (f) and (g),

$$a_r(\cos kl_r - \cosh kl_r) + c_r \sin kl_r + d_r \sinh kl_r = 0, \tag{h}$$

$$- a_r(\sin kl_r + \sinh kl_r) + c_r \cos kl_r + d_r \cosh kl_r = c_{r+1} + d_{r+1}, \tag{k}$$

$$a_r(\cos kl_r + \cosh kl_r) + c_r \sin kl_r - d_r \sinh kl_r = 2a_{r+1}. \tag{l}$$

Adding and subtracting (h) and (l) we obtain

$$a_r \cos kl_r + c_r \sin kl_r = a_r; \cosh kl_r - d_r \sinh kl_r = a_{r+1}$$

from which, provided $\sin kl_r$ is not zero,

$$c_r = \frac{a_{r+1} - a_r \cos kl_r}{\sin kl_r}; \qquad d_r = \frac{- a_{r+1} + a_r \cosh kl_r}{\sinh kl_r} \tag{m}$$

and

$$c_r + d_r = a_r (\coth kl_r - \cot kl_r) - a_{r+1}(\operatorname{cosech} kl_r - \operatorname{cosec} kl_r). \tag{n}$$

Using the notations:

$$\coth kl_r - \cot kl_r = \varphi_r, \tag{o}$$

$$\operatorname{cosech} kl_r - \operatorname{cosec} kl_r = \psi_r,$$

we obtain

$$c_r + d_r = a_r\varphi_r - a_{r+1}\psi_r.$$

In the same manner for the span $r + 1$

$$c_{r+1} + d_{r+1} = a_{r+1}\varphi_{r+1} - a_{r+2}\psi_{r+1}. \tag{p}$$

Substituting (m) and (p) in eq. (k), we obtain

$$a_r\psi_r - a_{r+1}(\varphi_r + \varphi_{r+1}) + a_{r+2}\psi_{r+1} = 0. \tag{q}$$

Writing an analogous equation for each intermediate support the following system of $(n - 1)$ equations will be obtained:

$$\begin{aligned} - a_2(\varphi_1 + \varphi_2) + a_3\psi_2 &= 0, \\ a_2\psi_2 - a_3(\varphi_2 + \varphi_3) + a_4\psi_3 &= 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{n-1}\psi_{n-1} - a_n(\varphi_{n-1} + \varphi_n) &= 0. \end{aligned} \tag{r}$$

Proceeding in the usual manner and putting equal to zero the determinant of these equations the frequency equation for the vibration of continuous beams will be obtained.

Take, for instance, a bar on three supports, then only one equation of the system (r) remains and the *frequency equation* will be

$$\varphi_1 + \varphi_2 = 0$$

or

$$\varphi_1 = -\varphi_2. \tag{s}$$

The frequencies of the consecutive modes of vibration will be obtained from the condition,

$$\varphi(kl_1) = -\varphi(kl_2).$$

In the solution of this transcendental equation it is convenient to draw a graph of the functions φ and $-\varphi$. In Fig. 181 φ and $-\varphi$ are given as functions of the argument kl expressed in degrees. The problem then reduces to finding by trial and error a line parallel to the x axis which cuts the graphs of φ and $-\varphi$ in points whose abscissae are in the ratio of the lengths of the spans.

Taking, for instance, $l_1 : l_2 = 6 : 4.5$, we obtain for the smallest root

$$kl_1 = 3.416,$$

from which the frequency of the fundamental mode of vibration becomes

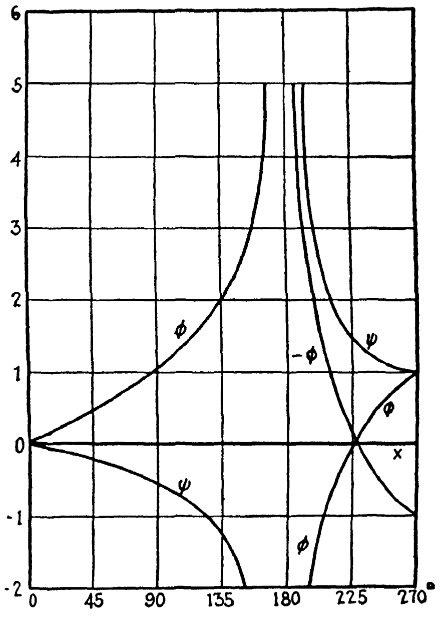

FIG. 181.

$$f_1 = \frac{k_1^2 a}{2\pi} = \frac{3.416^2}{2\pi l_1^2}\sqrt{\frac{EIg}{A\gamma}}.$$

For the next higher frequency we obtain

$$kl_1 = 4.787.$$

The third frequency is given approximately by $kl_1 = 6.690$ so that the consecutive frequencies are in the ratio 1 : 1.96 : 3.82. If the lengths of the spans tend to become equal it is seen from Fig. 181 that the smallest root tends to $kl_1 = kl_2 = \pi$. In the case of the fundamental type of vibration each span will be in the condition of a bar with hinged ends. Another type of vibration will be obtained by assuming the tangent at the intermediate support to be horizontal, then each span will be in the condition of a bar with one end built in and another simply supported.

In the case of three spans we obtain, from (r),

$$\begin{aligned} -a_2(\varphi_1 + \varphi_2) + a_3\psi_2 &= 0, \\ a_2\psi_2 - a_3(\varphi_2 + \varphi_3) &= 0, \end{aligned}$$

and the frequency equation becomes

$$(\varphi_1 + \varphi_2)(\varphi_2 + \varphi_3) - \psi_2^2 = 0. \tag{t}$$

Having tables of the functions φ and ψ * the frequency of the fundamental mode can be found, from (t), by a process of trial and error.

58. Forced Vibration of a Beam with Supported Ends.—*General.*—In the case of a beam with supported ends the general expression for flexural vibration is given by eq. (146). By using the symbols q_i for the generalized coordinates we obtain from the above equation

$$y = \sum_{i=1}^{i=\infty} q_i \sin \frac{i\pi x}{l}. \tag{a}$$

The expressions for the potential and kinetic energy will now be found from eqs. (130) and (131) by substituting $\sin i\pi x/l$ for X_i:

$$V = \frac{EI}{2} \sum_{i=1}^{i=\infty} q_i^2 \int_0^l \frac{i^4\pi^4}{l^4} \sin^2 \frac{i\pi x}{l}\, dx = \frac{EI\pi^4}{4l^3} \sum_{i=1}^{i=\infty} i^4 q_i^2 \tag{152}$$

$$T = \frac{A\gamma}{2g} \sum_{i=1}^{i=\infty} \dot{q}_i^2 \int_0^l \sin^2 \frac{i\pi x}{l}\, dx = \frac{A\gamma l}{4g} \sum_{i=1}^{i=\infty} \dot{q}_i^2. \tag{153}$$

If disturbing forces are acting on the beam, Lagrange's eq. (74) for any coordinate q_i will be

$$\frac{A\gamma l}{2g} \ddot{q}_i + \frac{EI\pi^4 i^4}{2l^3} q_i = Q_i$$

or

$$\ddot{q}_i + \frac{i^4\pi^4 a^2}{l^4} q_i = \frac{2g}{A\gamma l} Q_i, \tag{b}$$

* Such tables are given in the paper by E. R. Darnley; loc. cit., p. 345. Another method by using nomographic solution is given in the paper by D. M. Smith, loc. cit., p. 345, in which the application of this problem to the vibration of condenser tubes is shown.

in which Q_i denotes the generalized force corresponding to the coordinate q_i and a^2 is given by eq. (124). The general solution of eq. (*b*) is

$$q_i = A_i \cos \frac{i^2\pi^2 at}{l^2} + B_i \sin \frac{i^2\pi^2 at}{l^2}$$

$$+ \frac{l^2}{i^2\pi^2 a} \frac{2g}{A\gamma l} \int_0^t Q_i \sin \frac{i^2\pi^2 a(t - t_1)}{l^2} dt_1. \qquad (c)$$

The first two terms in this solution represent the free vibration determined by the initial conditions while the third term represents the vibration produced by the disturbing forces.

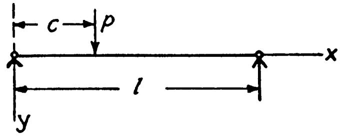

FIG. 182.

Pulsating Force.—As an example let us consider now the case of a pulsating force $P = P_0 \sin \omega t_1$ applied at a distance c from the left support (see Fig. 182). In order to obtain a generalized force Q_i assume that a small increase δq_i is given to a coordinate q_i. The corresponding deflection of the beam, from eq. (*a*), will be

$$\delta y = \delta q_i \sin \frac{i\pi x}{l}$$

and the work done by the external force P on this displacement is

$$P\delta q_i \sin \frac{i\pi c}{l}.$$

Then,

$$Q_i = P \sin \frac{i\pi c}{l} = P_0 \sin \frac{i\pi c}{l} \sin \omega t_1. \qquad (d)$$

Substituting in eq. (*c*) and considering only that part of the vibrations produced by the pulsating force we obtain

$$q_i = \frac{2g}{A\gamma} P_0 \sin \frac{i\pi c}{l} \left(\frac{l^3}{i^4\pi^4 a^2 - \omega^2 l^4} \sin \omega t \right.$$

$$\left. - \frac{\omega l^5}{i^2\pi^2 a(i^4\pi^4 a^2 - \omega^2 l^4)} \sin \frac{i^2\pi^2 at}{l^2} \right). \qquad (e)$$

Substituting in eq. (a), we have

$$y = \frac{2gP_0l^3}{A\gamma}\sum_{i=1}^{i=\infty}\frac{\sin\frac{i\pi c}{l}\sin\frac{i\pi x}{l}}{i^4\pi^4a^2 - \omega^2l^4}\sin\omega t$$

$$- \frac{2g\omega P_0l^5}{A\gamma\pi^2a}\sum_{i=1}^{i=\infty}\frac{\sin\frac{i\pi c}{l}\sin\frac{i\pi x}{l}}{i^2(i^4\pi^4a^2 - \omega^2l^4)}\sin\frac{i^2\pi^2at}{l^2}. \tag{154}$$

It is seen that the first series in this solution is proportional to $\sin \omega t$. It has the same period as the disturbing force and represents *forced vibrations* of the beam. The second series represents *free vibrations* of the beam produced by application of the force. These latter vibrations due to various kinds of resistance will be gradually damped out and only the forced vibrations, given by equation

$$y = \frac{2gP_0l^3}{A\gamma}\sum_{i=1}^{i=\infty}\frac{\sin\frac{i\pi c}{l}\sin\frac{i\pi x}{l}}{i^4\pi^4a^2 - \omega^2l^4}\sin\omega t, \tag{f}$$

are of practical importance.

If the pulsating force P is varying very slowly, ω is a very small quantity and ω^2l^4 can be neglected in the denominator of the series (f), then

$$y = \frac{2gPl^3}{A\gamma\pi^4a^2}\sum_{i=1}^{i=\infty}\frac{1}{i^4}\sin\frac{i\pi c}{l}\sin\frac{i\pi x}{l} \tag{j}$$

or, by using eq. (124),

$$y = \frac{2Pl^3}{EI\pi^4}\sum_{i=1}^{i=\infty}\frac{1}{i^4}\sin\frac{i\pi c}{l}\sin\frac{i\pi x}{l}. \tag{g}$$

This expression represents the statical deflection of the beam produced by the load P.* In the particular case, when the force P is applied at the middle, $c = l/2$ and we obtain

$$y = \frac{2Pl^3}{EI\pi^4}\left(\sin\frac{\pi x}{l} - \frac{1}{3^4}\sin\frac{3\pi x}{l} + \frac{1}{5^4}\sin\frac{5\pi x}{l} - \cdots\right). \tag{h}$$

The series converges rapidly and a satisfactory approximation for the

* See "Applied Elasticity," p. 131; "Strength of Materials," Vol. 2, p. 417.

deflections will be obtained by taking the first term only. In this manner we find for the deflection at the middle:

$$(y)_{x=\frac{l}{2}} = \frac{2Pl^3}{EI\pi^4} = \frac{Pl^3}{48.7EI}.$$

The error of this approximation is about 1.5 per cent.

Denoting by α the ratio of the frequency of the disturbing force to the frequency of the fundamental type of free vibration, we obtain

$$\alpha = \frac{\omega l^2}{a\pi^2}$$

and the series (f), representing forced vibrations, becomes

$$y = \frac{2P_0 \sin \omega t l^3}{EI\pi^4} \sum_{i=1}^{i=\infty} \frac{\sin \frac{i\pi c}{l} \sin \frac{i\pi x}{l}}{i^4 - \alpha^2}.$$

If the pulsating force is applied at the middle, we obtain

$$y = \frac{2P_0 \sin \omega t l^3}{EI\pi^4} \left(\frac{\sin \frac{\pi x}{l}}{1 - \alpha^2} - \frac{\sin \frac{3\pi x}{l}}{3^4 - \alpha^2} + \frac{\sin \frac{5\pi x}{l}}{5^4 - \alpha^2} \cdots \right). \tag{k}$$

For small α the first term of this series represents the deflection with good accuracy and comparing (k) with (h) it can be concluded that the ratio of the dynamical deflection to the statical deflection is approximately equal to

$$\frac{y_d}{y_s} = \frac{1}{1 - \alpha^2}. \tag{l}$$

If, for instance, the frequency of the disturbing force is four times as small as the frequency of the fundamental mode of vibration, the dynamical deflection will be about 6 per cent greater than the statical deflection.

Due to the fact that the problems on vibration of bars are represented by *linear* differential equations, the *principle of superposition* holds and if there are several pulsating forces acting on the beam, the resulting vibration will be obtained by superimposing the vibrations produced by the individual forces. The case of continuously distributed pulsating forces also can be solved in the same manner; the summation only has to be replaced by an integration along the length of the beam. Assume, for instance, that the beam is loaded by a uniformly distributed load of the intensity:

$$w = w_0 \sin \omega t.$$

Such a load condition exists, for instance, in a locomotive side rod under the action of lateral inertia forces. In order to determine the vibrations $w_0\, dc$ should be substituted for P_0 in eq. (f) and afterwards this equation should be integrated with respect to c within the limits $c = 0$ and $c = l$. In this manner we obtain

$$y = \frac{4gw_0 l^4}{A\gamma\pi} \sum_{i=1}^{i=\infty} \frac{\sin \frac{i\pi x}{l}}{i(i^4\pi^4 a^2 - \omega^2 l^4)} \sin \omega t. \tag{m}$$

If the frequency of the load is very small in comparison with the frequency of the fundamental mode of vibration of the bar the term $\omega^2 l^4$ in the denominators of the series (m) can be neglected and we obtain,

$$y = \frac{4wl^4}{EI\pi^5}\left(\frac{\sin \frac{\pi x}{l}}{1^5} + \frac{\sin \frac{3\pi x}{l}}{3^5} + \frac{\sin \frac{5\pi x}{l}}{5^5} + \cdots\right). \tag{n}$$

This very rapidly converging series represents the statical deflection of the beam produced by a uniformly distributed load w. By taking $x = l/2$ we obtain for the deflection at the middle

$$(y)_{x=\frac{l}{2}} = \frac{4wl^4}{EI\pi^5}\left(1 - \frac{1}{3^5} + \frac{1}{5^5} - \cdots\right). \tag{p}$$

If only the first term of this series be taken, the error in the deflection at the middle will be about 1/4 per cent. If the frequency of the pulsating load is not small enough to warrant application of the statical equation, the same method can be used as was shown in the case of a single force and we will arrive at the same conclusion as represented by eq. (l).

Moving Constant Force.—If a constant vertical force P is moving along the length of a beam it produces vibrations which can be calculated without any difficulty by using the general eq. (c). Let v denote the constant * velocity of the moving force and let the force be at the left support at the initial moment ($t = 0$), then at any other moment $t = t_1$ the distance of this force from this left support will be vt_1. In order to

* The case when the velocity is not constant has been discussed by A. N. Lowan, Phil. Mag. Ser. 7, Vol. 19, p. 708, 1935.

determine the generalized force Q_i for this case assume that the coordinate q_i in the general expression (a) of the deflection curve obtains an infinitely small increase δq_i. The work done by the force P due to this displacement will be

$$P(\delta y)_{x=vt_1} = P\delta q_i \sin\frac{i\pi v t_1}{l}.$$

Hence the generalized force

$$Q_i = P\sin\frac{i\pi v t_1}{l}.$$

Substituting this in the third term of equation (c) the following expression will be found for the vibrations produced by the moving load.*

$$y = \frac{2gPl^3}{A\gamma\pi^2}\sum_{i=1}^{i=\infty}\frac{\sin\dfrac{i\pi x}{l}}{i^2(i^2\pi^2a^2 - v^2l^2)}\sin\frac{i\pi vt}{l}$$
$$- \frac{2gPl^4v}{A\gamma\pi^3a}\sum_{i=1}^{i=\infty}\frac{\sin\dfrac{i\pi x}{l}}{i^3(i^2\pi^2a^2 - v^2l^2)}\sin\frac{i^2\pi^2at}{l^2}. \qquad (155)$$

The first series in this solution represents forced vibrations and the second series free vibrations of the beam.

If the velocity v of the moving force be very small, we can put $v = 0$ and $vt = c$ in the solution above; then

$$y = \frac{2gPl^3}{A\gamma\pi^4a^2}\sum_{i=1}^{i=\infty}\frac{1}{i^4}\sin\frac{i\pi c}{l}\sin\frac{i\pi x}{l}.$$

This is the statical deflection of the beam produced by the load P applied at the distance c from the left support (see eq. (j)). By using the notation,

$$\alpha^2 = \frac{v^2l^2}{a^2\pi^2}, \qquad (q)$$

* This problem is of practical interest in connection with the study of bridge vibrations. The first solution of this problem was given by A. N. Kryloff; see Mathematische Annalen, Vol. 61 (1905). See also writer's paper in the "Bulletin of the Polytechnical Institute in Kiev" (1908). (German translation in Zeitschr. f. Math. u. Phys., Vol. 59 (1911)). Prof. C. E. Inglis in the Proc. of The Inst. of Civil Engineers, Vol. 218 (1924), London, came to the same results. If instead of moving force a moving weight is acting on the beam, the problem becomes more complicated. See H. H. Jeffcott, Phil. Mag. 7 ser., Vol. 8, p. 66, 1929, and H. Steuding, Ingenieur-Archiv, Vol. 5, p. 275, 1934.

the forced vibrations in the general solution (155) can be presented in the following form

$$y = \frac{2Pl^3}{EI\pi^4} \sum_{i=1}^{i=\infty} \frac{\sin \frac{i\pi x}{l} \sin \frac{i\pi vt}{l}}{i^2(i^2 - \alpha^2)} \cdot \tag{r}$$

It is interesting to note that this deflection completely coincides with the statical deflection of a beam* on which in addition to the lateral load P applied at a distance $c = vt$ from the left support a longitudinal compressive force S is acting, such that

$$\frac{S}{S_{cr}} = \frac{Sl^2}{EI\pi^2} = \alpha^2. \tag{s}$$

Here S_{cr} denotes the known *critical* or *column load* for the beam.

From the eqs. (s) and (q) we obtain

$$\frac{Sl^2}{EI\pi^2} = \frac{v^2l^2}{a^2\pi^2}$$

or

$$S = \frac{v^2A\gamma}{g}. \tag{t}$$

The effect of this force on the statical deflection of the beam loaded by P is equivalent to the effect of the velocity of a moving force P on the deflection (r) representing forced vibrations.

By increasing the velocity v, a condition can be reached where one of the denominators in the series (155) becomes equal to zero and resonance takes place. Assume, for instance, that

$$a^2\pi^2 = v^2l^2. \tag{u}$$

In this case the period of the fundamental vibration of the beam, equal to $2l^2/a\pi$, becomes equal to $2l/v$ and is twice as great as the time required for the force P to pass over the beam. The denominators in the first terms of both series in eq. (155) become, under the condition (u), equal to

* See "Applied Elasticity," p. 163. By using the known expression for the statical deflection curve the finite form of the function, from which the series (r) has its origin, can be obtained.

ːro and the sum of these two terms will be

$$\frac{2gPl^3}{A\gamma\pi^2}\sin\frac{\pi x}{l}\,\frac{\sin\dfrac{\pi vt}{l} - \dfrac{lv}{\pi a}\sin\dfrac{\pi^2 at}{l^2}}{\pi^2a^2 - v^2l^2}.$$

his has the form 0/0 and can be presented in the usual way as follows
ee p. 17):

$$-\frac{Pg}{\gamma A\pi v}\,t\cos\frac{\pi vt}{l}\sin\frac{\pi x}{l} + \frac{Pgl}{\gamma A\pi^2v^2}\sin\frac{\pi vt}{l}\sin\frac{\pi x}{l}. \qquad (v)$$

This expression has its maximum value when

$$t = \frac{l}{v}$$

ıd is then equal to

$$\frac{Pgl}{\gamma A\pi^2v^2}\left(\sin\frac{\pi vt}{l} - \frac{\pi vt}{l}\cos\frac{\pi vt}{l}\right)_{t=l/v}\sin\frac{\pi x}{l} = \frac{Pl^3}{EI\pi^3}\sin\frac{\pi x}{l}. \qquad (w)$$

Taking into consideration that the expression (v) represents a satisıctory approximation for the dynamical deflection given by equation 155) it can be concluded that the maximum dynamical deflection at the ːsonance condition (u) is about 50 per cent greater than the maximum ,atical deflection which is equal to

$$\frac{Pl^3}{48EI}.$$

It is interesting to note that the maximum dynamical deflection occurs hen the force P is leaving the beam. At this moment the deflection under ıe force P is equal to zero, hence the work one by this force during the passing of the eam is also equal to zero. In order to explain ıe source of the energy accumulated in the ibrating beam during the passing over of the ɔrce P we should assume that there is no fricon and the beam produces a reaction R in the irection of the normal (Fig. 183). In this case, ·om the condition of equilibrium it follows that there should exist a orizontal force, equal to $P(dy/dx)$. The work done by this force during ,s passage along the beam will be

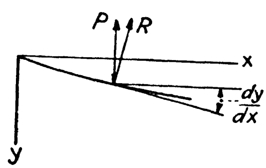

FIG. 183.

$$E = -\int_0^{l/v} P\left(\frac{dy}{dx}\right)_{x=vt} vdt.$$

Substituting expression (v) for y we obtain

$$E = -\frac{P^2 g}{\gamma A \pi v^2}\int_0^{l/v}\left(\sin\frac{\pi vt}{l} - \frac{\pi vt}{l}\cos\frac{\pi vt}{l}\right)\cos\frac{\pi vt}{l}\,vdt = \frac{P^2 gl}{\gamma A \pi^2 v^2}\cdot\frac{\pi^2}{4}$$

or, by taking into consideration eqs. (u) and (124) we obtain

$$E = \frac{\pi^2}{4}\frac{P^2 l^3}{EI\pi^4}.$$

This amount of work is very close* to the amount of the potential energy of bending in the beam at the moment $t = l/v$.

In the case of bridges, the time it takes to cross the bridge is usually large in comparison with the period of the fundamental type of vibration and the quantity α^2, given by eq. (q), is small. Then by taking only the first term in each series of eq. (155) and assuming that in the most unfavorable case the amplitudes of the forced and free vibrations are added to one another, we obtain for the maximum deflection,

$$\begin{aligned} y_{\max} &= \frac{2gPl^3}{\gamma A \pi^2}\left(\frac{1}{\pi^2 a^2 - v^2 l^2} + \frac{vl}{a\pi}\frac{1}{\pi^2 a^2 - v^2 l^2}\right) \\ &= \frac{2Pl^3}{EI\pi^4}\frac{1+\alpha}{1-\alpha^2} = \frac{2Pl^3}{EI\pi^4}\frac{1}{1-\alpha}. \end{aligned} \tag{156}$$

This is a somewhat exaggerated value of the maximum dynamical deflection, because damping was completely neglected in the above discussion.

By using the principle of superposition the solution of the problem in the case of a system of concentrated moving forces and in the case of moving distributed forces can be also solved without difficulty.†

Moving Pulsating Force.‡—Consider now the case when a pulsating force is moving along the beam with a constant velocity v. Such a condition may occur, for instance, when an imperfectly balanced locomotive passes over a bridge (Fig. 184). The vertical component of the centrifugal force

* The potential energy of the beam bent by the force P at the middle is

$$V = \frac{P^2 l^3}{96EI} \quad \text{and} \quad \frac{E}{V} = 2.43.$$

This ratio is very close to the square of the ratio of the maximum deflections for the dynamical and statical conditions which is equal to $(48/\pi^3)^2 = 2.38$. The discrepancy should be attributed to the higher harmonics in the deflection curve.

† See writer's paper mentioned above.

‡ See writer's paper in Phil. Mag., Vol. 43 (1922), p. 1018.

P*, due to the unbalance, is $P \cos \omega t_1$, where ω is the angular velocity of the driving wheel. By using the same manner of reasoning as before, the following expression for the generalized force, corresponding to the generalized coordinate q_i will be obtained.

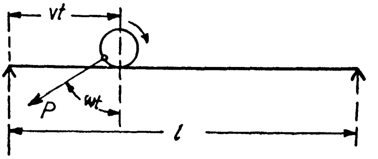

FIG. 184.

$$Q_i = P \cos \omega t_1 \sin \frac{i\pi v t_1}{l}.$$

Substituting this in the third term of the general solution (c), we obtain

$$y = \frac{Pl^3}{EI\pi^4} \sum_{i=1}^{i=\infty} \sin \frac{i\pi x}{l} \left\{ \frac{\sin\left(\frac{i\pi v}{l} + \omega\right)t}{i^4 - (\beta + i\alpha)^2} + \frac{\sin\left(\frac{i\pi v}{l} - \omega\right)t}{i^4 - (\beta - i\alpha)^2} - \frac{\alpha}{i}\left(\frac{\sin \frac{i^2\pi^2 a t}{l^2}}{-i^2\alpha^2 + (i^2 - \beta)^2} + \frac{\sin \frac{i^2\pi^2 a t}{l^2}}{-i^2\alpha^2 + (i^2 + \beta)^2} \right) \right\} \tag{157}$$

where $\alpha = vl/\pi a$ is the ratio of the period $\tau = 2l^2/\pi a$ of the fundamental type of vibration of the beam to twice the time, $\tau_1 = l/v$, it takes the force P to pass over the beam,

$\beta = \tau/\tau_2$ is the ratio of the period of the fundamental type of vibration of the beam to the period $\tau_2 = 2\pi/\omega$ of the pulsating force.

When the period τ_2 of the pulsating force is equal to the period τ of the fundamental type of vibration of the beam $\beta = 1$ and we obtain the condition of resonance. The amplitude of the vibration during motion of the pulsating force will be gradually built up and attains its maximum at the moment $t = l/v$ when the first term (for $i = 1$) in the series on the right of (157), which is the most important part of y, may be reduced to the form

$$\frac{1}{\alpha}\frac{2Pl^3}{EI\pi^4} \sin \frac{\pi x}{l} \sin \omega t$$

and the maximum deflection is given by the formula

$$\delta_{\max} = \frac{1}{\alpha}\frac{2Pl^3}{EI\pi^4} = \frac{2\tau_1}{\tau} \cdot \frac{2Pl^3}{EI\pi^4}. \tag{158}$$

* It is assumed that at the initial moment $t_1 = 0$ the centrifugal force is acting in downwards direction.

Due to the fact that in actual cases the time interval $\tau_1 = l/v$ is large in comparison with the period τ of the natural vibration, the maximum dynamical deflection produced by the pulsating force P will be many times greater than the deflection $2Pl^3/EI\pi^4$, which would be produced by the same force if applied statically at the middle of the beam. Some applications of eq. (158) for calculating the impact effect on bridges will be given in the next article.

59. Vibration of Bridges.—It is well known that a rolling load produces in a bridge or in a girder a greater deflection and greater stresses than the same load acting statically. Such an "*impact effect*" of live loads on bridges is of great practical importance and many engineers have worked on the solution of this problem.* There are various causes producing impact effect on bridges of which the following will be discussed: (1) Live-load effect of a smoothly-running load; (2) Impact effect of the balance-weights of the locomotive driving wheels and (3) Impact effect due to irregularities of the track and flat spots on the wheels.

Live-load Effect of a Smoothly Running Mass.—In discussing this problem two extreme cases will be considered: (1) when the mass of the moving load is large in comparison with the mass of the beam, i.e., girder or rail bearer, and (2) when the mass of the moving load is small in comparison with the mass of the bridge. In the first case the mass of the beam can be neglected. Then the deflection of the beam under the load at any position of this load will be proportional to the pressure R, which the rolling load P produces on the beam (Fig. 185) and can be calculated from the known equation of statical deflection:

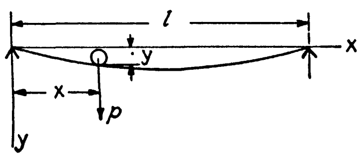

FIG. 185.

$$y = \frac{Rx^2(l - x)^2}{3lEI}. \tag{a}$$

In order to obtain the pressure R the inertia force $-(P/g)(d^2y/dt^2)$ should be added to the rolling load P. Assuming that the load is moving along the beam with a constant velocity v, we obtain

$$\frac{dy}{dt} = v\frac{dy}{dx}; \qquad \frac{d^2y}{dt^2} = v^2\frac{d^2y}{dx^2}$$

* The history of the subject is extensively discussed in the famous book by Clebsch' Theorie der Elastizität fester Korper, traduite p. S. Venant (Paris 1883), see Note du par. 61, p. 597.

and the pressure on the beam will be

$$R = P\left(1 - \frac{v^2}{g}\frac{d^2y}{dx^2}\right) \tag{b}$$

Substituting in eq. (a) we obtain

$$y = P\left(1 - \frac{v^2}{g}\frac{d^2y}{dx^2}\right)\frac{x^2(l-x)^2}{3lEI}. \tag{159}$$

This equation determines the path of the point of contact of the rolling load with the beam.* An approximation of the solution of eq. (159) will be obtained by assuming that the path is the same as at zero speed ($v = 0$) and by substituting

$$\frac{Px^2(l-x)^2}{3lEI}$$

for y in the right side of this equation. Then by simple calculations it can be shown that y becomes maximum when the load is at the middle of the span and the maximum pressure will be

$$R_{\max} = P\left(1 + \frac{v^2}{g}\frac{Pl}{3EI}\right). \tag{c}$$

The maximum deflection in the center of the beam increases in the same rate as the pressure on it, so that:

$$\delta_d = \delta_{st}\left(1 + \frac{v^2}{g}\frac{Pl}{3EI}\right). \tag{160}$$

This approximate solution as compared with the result of an exact solution of the eq. (159)† is accurate enough for practical applications. The additional term in the brackets is usually very small and it can be con-

* This equation was established by Willis: Appendix to the Report of the Commissioners . . . to inquire into the Application of Iron to Railway Structures (1849), London. This report was reprinted in the "Treatise on the Strength of Timber, Cast and Malleable Iron," by P. Barlow, 1851, London.

† The exact solution of eq. (159) was obtained by G. G. Stokes, see Math. and Phys. Papers, Vol. 2, p. 179. The same problem has been discussed also by H. Zimmermann, see "Die Schwingungen eines Traegers mit bewegter Last." Berlin, 1896. It should be noted that the integration of eq. (159) can be made also numerically by using the method explained before, see p. 126. In this manner solutions for a beam on elastic supports and for continuous beams were obtained by Prof. N. P. Petroff, see, the Memoirs of The Russian Imperial Technical Society (1903).

cluded that the "live-load effect" in the case of small girders has no practical importance.

In the second case when the mass of the load is small in comparison with the mass of the bridge the moving load can be replaced, with sufficient accuracy, by a moving force and then the results given in article 58 can be used. Assuming, for instance, that for three single track railway bridges with spans of 60 feet, 120 feet and 360 feet, the natural frequencies are as shown in the table below,*

	l = 60 ft.	120 ft.	360 ft.
f =	9	5	2 per sec.
$(\alpha)_{v=120 \text{ ft. per sec.}}$ =	1/9	1/10	1/12

and taking the velocity $v = 120$ feet per sec., the quantity α, representing the ratio of the period of the fundamental type of vibration to twice the time l/v for the load to pass over the bridge will be as shown in the third line of the table. Now on the basis of solution (156) it can be concluded † that for a span of 60 feet and with a very high velocity, the increase in deflection due to the live load effect is about 12 per cent and this is still diminished with a decrease of velocity and with an increase of span. If several moving loads are acting on the bridge the oscillations associated with these should be superimposed. Only in the exceptional case of synchronism of these vibrations the resultant live-load effect on the system will be equal to the sum of the effects of the separate loads and the increase in deflection due to this effect will be in the same proportion as for a single load. From these examples it can be concluded that the live-load effect of a smooth-running load is not an important factor and in the most unfavorable cases it will hardly exceed 10 per cent. Much more serious effects may be produced, as we will see, by pulsating forces due to rotating balance weights of steam locomotives.

Impact Effect of Unbalanced Weights.—The most unfavorable condition

* Some experimental data on vibrations of bridges can be found in the following papers: A. Buhler, Stosswirkungen bei eisernen Eisenbahnbrucken, Druckschrift zum Intern. Kongress fur Brückenbau, Zürich, 1926; W. Hort, Stossbeanspruchungen und Schwingungen . . . Die Bautechnik, 1928, Berlin, and in books N. Streletzky, "Ergebnisse der experimentellen Brückenuntersuchungen" Berlin, 1928, and C. E Inglis, "A Mathematical Treatise on Vibrations in Railway Bridges," Cambridge, 1934.

† The bridge is considered here as a simple beam of a constant cross section. Vibration of trusses has been discussed by H. Reissner, Zeitschr. f. Baut., Vol. 53 (1903), p. 135, E. Pohlhausen, Zeitschr. f. Angew. Math. u. Mech., Vol. 1 (1921), p. 28, and K. Federhofer, "Der Stahlbau." 1934, Heft 1.

occurs in the case of resonance when the number of revolutions per second of the driving wheels is equal to the frequency of natural vibration of the bridge. For a short span bridge the frequency of natural vibration is usually so high that synchronism of the pulsating load and the natural vibration is impossible at any practical velocity. By taking, for instance, six revolutions per second of the driving wheels as the highest limit and taking the frequencies of natural vibration from the table above it can be concluded that the resonance condition is hardly possible for spans less than 100 ft. For larger spans resonance conditions should be taken into consideration and the impact effect should be calculated from eq. (158).

Let P_1 be the maximum resultant pressure on the rail due to the counterweights when the driving wheels are revolving once per second.
n is the total number of revolutions of the driving wheels during passage along the bridge.

Then, from eq. (158), we obtain the following additional deflection due to the impact effect,

$$\delta_{\max} = \frac{2n}{\tau^2} \frac{2P_1 l^3}{EI\pi^4}. \tag{161}$$

We see that in calculating the impact effect due to unbalanced weights we have to take consideration of: (1) the statical deflection produced by the force P_1, (2) the period τ of the natural vibration of the bridge and (3) the number of revolutions n. All these quantities are usually disregarded in impact formulas as applied in bridge design.

In order to obtain some idea about the amount of this impact effect let us apply eq. (161) to a numerical example* of a locomotive crossing a bridge of 120 feet span. Assuming that the locomotive load is equivalent to a uniform load of 14,700 lbs. per linear foot distributed over a length of 15 feet, and that the train load following and preceding the locomotive is equivalent to a uniformly distributed load of 5,500 lbs. per linear foot, the maximum central deflection of each girder is $(2l^3/EI\pi^4)$ (275,000) approximately. The same deflection when the locomotive approaches the support and the train completely covers the bridge is $(2l^3/EI\pi^4)$ (206,000) approximately. Taking the number of revolutions $n = 8$ (the diameter of the wheels equal to 4 feet and 9 inches) and the maximum pulsating pressure on each girder at the resonance condition equal to $P_1/\tau^2 = 18{,}750$

* The figures below are taken from the paper by C. E. Inglis, previously mentioned (see p. 353).

lbs., the additional deflection, calculated from eq. (161), will be $(2l^3/EI\pi^4)$ (300,000). Adding this to the statical deflection, calculated above for the case of the locomotive approaching the end of the bridge, we obtain for the complete deflection at the center $(2l^3/EI\pi^4)$ (506,000). Comparing this with the maximum statical central deflection $(2l^3/EI\pi^4)$ $\times$ (275,000), given above, it can be concluded that the increase in deflection due to impact is in this case about 84 per cent. Assuming the number of revolutions n equal to 6 (the diameter of driving wheels equal to $6\frac{1}{2}$ feet) and assuming again a condition of resonance, we will obtain for the same numerical example an increase in deflection equal to 56 per cent.

In the case of bridges of shorter spans, when the frequency of natural vibration is considerably larger than the number of revolutions per second of the driving wheels, a satisfactory approximation can be obtained by taking only the first term in the series (157) and assuming the most unfavorable condition, namely, that $\sin\,([\pi v/l] + \omega)t$ and $\sin\,([\pi v/l] - \omega)t$ become equal to 1 and $\sin \pi^2 at/l^2$ equal to -1 at the moment $t = l/2v$ when the pulsating force arrives at the middle of the span. Then the additional deflection, from (157), will be

$$= \frac{Pl^3}{EI\pi^4}\left(\frac{1}{1-(\beta+\alpha)^2} + \frac{1}{1-(\beta-\alpha)^2} + \frac{\alpha}{(1-\beta)^2-\alpha^2} + \frac{\alpha}{(1+\beta)^2-\alpha^2}\right)$$

$$= \frac{2Pl^3}{EI\pi^4}\,\frac{1-\alpha}{(1-\beta[1+\alpha/\beta])(1+\beta[1-\alpha/\beta])}. \tag{162}$$

Consider, for instance, a 60-foot span bridge and assume the same kind of loading as in the previous example, then the maximum statical deflection is $(2l^3/EI\pi^4)$ (173,000) approximately. If the driving wheels have a circumference of 20 feet and make 6 revolutions per second, the maximum downwards force on the girder will be $18{,}750(6/5)^2 = 27{,}000$ lbs. Assuming the natural frequency of the bridge equal to 9, we obtain from eq. (153)

$$\delta = \frac{2l^3}{EI\pi^4}\,(27{,}000 \times 2.57) = \frac{2l^3}{EI\pi^4}\,(69{,}400).$$

Hence,

$$\frac{\text{dynamical deflection}}{\text{statical deflection}} = \frac{173 + 69.4}{173} = 1.40.$$

The impact effect of the balancing weights in this case amounts to 40 per cent.

In general it will be seen from the theory developed above that the most severe impact effects will be obtained in the shortest spans for which a resonance condition may occur (about 100 feet spans for the assumption made above) because in this case the resonance occurs when the pulsating disturbing force has its greatest magnitude. With increase in the span the critical speed decreases and also the magnitude of the pulsating load, consequently the impact effect decreases. For very large spans, when the frequency of the fundamental type of vibration is low, synchronism of the pulsating force with the second mode of vibration having a node at the middle of the span becomes theoretically possible and due to this cause an increase in the impact effect may occur at a velocity of about four times as great as the first critical speed.

It should be noted that all our calculations were based on the assumption of a pulsating force moving along the bridge. In actual conditions we have rolling *masses*, which will cause a variation in the natural frequency of the bridge in accordance with the varying position of the loads. This variability of the natural frequency which is especially pronounced in short spans is very beneficial because the pulsating load will no longer be in resonance all the time during passing over the bridge and its cumulative effect will not be as pronounced as is given by the above theory. From experiments made by the Indian Railway Bridge Committee,* it is apparent that on the average the maximum deflection occurs when the engine has traversed about two-thirds of the span and that the maximum impact effect amounts to only about one-thrid of that given by eq. (161). It should be noted also that the impact effect is proportional to the force P_1 and depends therefore on the type of engine and on the manner of balancing. While in a badly balanced two cylinder engine the force P_1 may amount to more than 1000 lbs.,† in electric locomotives, perfect balancing can be obtained without introducing a fluctuating rail pressure. This absence of impact effect may compensate for the increase in axle load in modern heavy electric locomotives.

In the case of short girders and rail bearers whose natural frequencies are very high, the effect of counter-weights on the deflection and stresses can be calculated with sufficient accuracy by neglecting vibrations and using the statical formula in which the centrifugal forces of the counter-

* See Bridge Sub-Committee Reports, 1925; Calcutta: Government of India Central Publication Branch, Technical Paper No. 247 (1926). Similar conclusions were obtained also by C. E. Inglis, see his book, "Vibrations in Bridges," 1934.

† Some data on the values of P_1 for various types of engines are given in the Bridge Sub-Committee Report, mentioned above.

weights should be added to the statical rail pressures. The effect of these centrifugal forces may be especially pronounced in the case of short spans when only a small number of wheels can be on the girder simultaneously.

Impact Effects Due to Irregularities of Track and Flats on Wheels.—Irregularities like low spots on the rails, rail joints, flats on the wheels, etc., may be responsible for considerable impact effect which may become especially pronounced in the case of short spans. If the shape of the low spots in the track or of the flats on the wheels is given by a smooth curve, the methods used before in considering the effect of road unevenness on the vibrations of vehicles (see p. 238) and the effect of low spots on deflection of rails (see p. 107) can also be applied here for calculating the additional pressure of the wheel on the rail. This additional pressure will be proportional to the unsprung mass of the wheel and to the square of the velocity of the train. It may attain a considerable magnitude and has practical importance in the case of short bridges and rail bearers. This additional dynamical effect produced by irregularities in the track and flats on the wheels justifies the high impact factor usually applied in the design of short bridges. By removing rail joints from the bridges and by using ballasted spans or those provided with heavy timber floors, the effect of these irregularities can be diminished and the strength condition considerably improved.

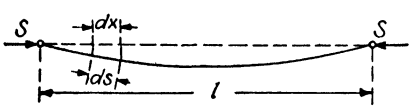

FIG. 186.

60. Effect of Axial Forces on Lateral Vibrations.—*Bar with Hinged Ends.*—As a first example of this kind of problems let us consider the case of a bar compressed by two forces S (see Fig. 186). The general expression for the lateral vibration will be the same as before (see eq. (146)).

$$y = \sum_{i=1}^{i=\infty} q_i \sin \frac{i\pi x}{l}. \tag{a}$$

The difference will be only in the expression for the potential energy of the system. It will be appreciated that during lateral deflection in this case not only the energy of bending but also the change in the energy of compression should be considered. Due to lateral deflection the initially compressed center line of the bar expands somewhat* and the potential energy of compression diminishes. The increase in length of the center

* The hinges are assumed immovable during vibration.

line will be (see Fig. 186),

$$\int_0^l (ds - dx) \approx \tfrac{1}{2}\int_0^l \left(\frac{dy}{dx}\right)^2 dx.$$

The corresponding diminishing of the energy of compression is *

$$\frac{S}{2}\int_0^l \left(\frac{dy}{dx}\right)^2 dx = \frac{S}{2}\int_0^l \left(\sum_{i=1}^{i=\infty} q_i \frac{i\pi}{l}\cos\frac{i\pi x}{l}\right)^2 dx = \frac{S\pi^2}{4l}\sum_{i=1}^{i=\infty} i^2 q_i^2. \qquad (b)$$

If the ends of the bar are free to slide in an axial direction eq. (*b*) will represent the work of forces S. For the energy of bending the equation (152) previously obtained will be used. Hence the complete potential energy becomes

$$V = \frac{EI\pi^4}{4l^3}\sum_{i=1}^{i=\infty} i^4 q_i^2 - \frac{S\pi^2}{4l}\sum_{i=1}^{i=\infty} i^2 q_i^2. \qquad (163)$$

The kinetic energy of the bar, from eq. (153) is

$$T = \frac{\gamma A l}{4g}\sum_{i=1}^{i=\infty} \dot{q}_i^2, \qquad (164)$$

and Lagrange's equation for any coordinate q_i will be

$$\frac{\gamma A l}{2g}\ddot{q}_i + \frac{EI\pi^4}{2l^3}\left(i^4 - \frac{Sl^2}{EI\pi^2} i^2\right) q_i = 0.$$

By using the notations,

$$a^2 = \frac{EIg}{\gamma A}, \qquad \alpha^2 = \frac{Sl^2}{EI\pi^2}, \qquad (165)$$

we obtain

$$\ddot{q}_i + \frac{a^2\pi^4}{l^4}(i^4 - \alpha^2 i^2) q_i = 0,$$

from which,

$$q_i = C\cos\left(\frac{a\pi^2 i^2}{l^2}\sqrt{1 - \frac{\alpha^2}{i^2}}\right)t + D\sin\left(\frac{a\pi^2 i^2}{l^2}\sqrt{1 - \frac{\alpha^2}{i^2}}\right)t. \qquad (166)$$

Substituting this in (*a*) the complete expression for free vibrations will be obtained.

Comparing this solution (166) with (143) it can be concluded that,

* Only those deflections are considered here which are so small that any change in longitudinal force can be neglected.

due to the compressive force S, the frequencies of natural vibration are diminished in the proportion

$$\sqrt{1 - \frac{\alpha^2}{i^2}}.$$

If α^2 approaches 1, the frequency of the fundamental type of vibration approaches zero, because at this value of α^2 the compressive force S attains its critical value $EI\pi^2/l^2$ at which the straight form of equilibrium becomes unstable and the bar buckles sidewise.

If instead of a compressive a tensile force S is acting on the bar the frequency of vibration increases. In order to obtain the free vibrations in this case it is only necessary to change the sign of α^2 in eq. (166). Then

$$q_i = C \cos\left(\frac{a\pi^2 i^2}{l^2}\sqrt{1 + \frac{\alpha^2}{i^2}}\right) t + D \sin\left(\frac{a\pi^2 i^2}{l^2}\sqrt{1 + \frac{\alpha^2}{i^2}}\right) t. \tag{167}$$

When α^2 is a very large number (such conditions can be obtained with thin wires or strings) 1 can be neglected in comparison with α^2/i^2 and we obtain from (167)

$$q_i = C \cos\frac{i\pi}{l}\sqrt{\frac{gS}{A\gamma}}\, t + D \sin\frac{i\pi}{l}\sqrt{\frac{gS}{A\gamma}}\, t.$$

Substituting in (a)

$$y = \sum_{i=1}^{i=\infty} \sin\frac{i\pi x}{l}\left(C \cos\frac{i\pi}{l}\sqrt{\frac{gS}{A\gamma}}\, t + D \sin\frac{i\pi}{l}\sqrt{\frac{gS}{A\gamma}}\, t\right). \tag{168}$$

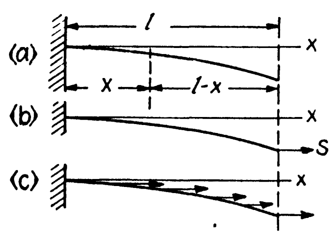

FIG. 187.

This is the general solution for the lateral vibrations of a stretched string where the rigidity of bending is neglected.

Cantilever Beam.—In this case only an approximate solution, by using the Rayleigh method, will be given. As a basis of this calculation the deflection curve

$$y = \frac{w}{2EI}\left(\frac{1}{2}l^2x^2 - \frac{1}{3}lx^3 + \frac{1}{12}x^4\right) \tag{c}$$

of a cantilever under the action of its weight w per unit length will be taken. The potential energy of bending in this case is

$$V = \frac{EI}{2}\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx = \frac{w^2l^5}{40EI}. \tag{d}$$

If the deflection during vibration is given by $y \cos pt$, the maximum kinetic energy of vibration will be

$$T = \frac{wp^2}{2g}\int_0^l y^2 dx = \frac{w^3p^2l^9}{E^2I^2g}\,\frac{13}{8\cdot 9\cdot 90}. \tag{e}$$

Putting (d) equal to (e) the following expression for the frequency and the period of vibration of a cantilever (Fig. 187a) will be obtained

$$f = \frac{p}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{90\times 9}{65}}\sqrt{\frac{EIg}{wl^4}}, \tag{169}$$

$$\tau = \frac{1}{f} = 2\pi\sqrt{\frac{65}{90\times 9}}\sqrt{\frac{wl^4}{EIg}} = \frac{2\pi}{3.530}\sqrt{\frac{wl^4}{EIg}}. \tag{170}$$

The error of this approximate solution is less than ½ per cent (see p. 344).

In order to calculate the frequency when a tensile force S is acting at the end of the cantilever, Fig. 187b, the quantity

$$\frac{S}{2}\int_0^l \left(\frac{dy}{dx}\right)^2 dx = \frac{Sw^2l^7}{8\times 14\times E^2I^2}, \tag{f}$$

which is equal and opposite in sign to the work done by the tensile force S during bending, should be added to the potential energy of bending, calculated above (eq. (d)). Then

$$V = \frac{w^2l^5}{40EI}\left(1 + \frac{5}{14}\frac{Sl^2}{EI}\right). \tag{g}$$

Due to this increase in potential energy the frequency of vibration will be found by multiplying the value (169) by

$$\sqrt{1 + \frac{5}{14}\frac{Sl^2}{EI}}. \tag{171}$$

It is interesting to note that the term $5/14\ Sl^2/EI$ differs only about 10 per cent from the quantity $\alpha^2 = 4Sl^2/EI\pi^2$, representing the ratio of the longitudinal force S to the critical column force for a cantilever.

If tensile forces s are uniformly distributed along the length of the cantilever (Fig. 187c), the term to be added to the energy of bending will be

$$\tfrac{1}{2}\int_0^l \left(\frac{dy}{dx}\right)^2 s(l - x)dx = \frac{w^2l^7}{8\times 14E^2I^2}\,\frac{7sl}{20}. \tag{172}$$

Comparing with eq. (f) it can be concluded that the effect on the frequency of uniformly distributed tensile forces is the same as if 7/20 of the sum of these forces be applied at the end of the cantilever.

This result may be of some practical interest in discussing the effect of the centrifugal force on the frequency of vibration of turbine blades (see p. 382).

61. Vibration of Beams on Elastic Foundation.—Assume that a beam with hinged ends is supported along its length by a *continuous elastic foundation*, the rigidity of which is given by the magnitude k of the *modulus of foundation*. k is the load per unit length of the beam necessary to produce a compression in the foundation equal to unity. If the mass of the foundation can be neglected the vibrations of such a beam can easily be studied by using the same methods as before. It is only necessary in calculating the potential energy of the system to add to the energy of bending of the beam, the energy of deformation of the elastic foundation. Taking, as before, for hinged ends,

$$y = \sum_{i=1}^{i=\infty} q_i \sin \frac{i\pi x}{l}, \tag{a}$$

we obtain

$$V = \frac{EI}{2}\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx + \frac{k}{2}\int_0^l y^2 dx = \frac{EI\pi^4}{4l^3}\sum_{i=1}^{i=\infty} i^4 q_i^2 + \frac{kl}{4}\sum_{i=1}^{i=\infty} q_i^2 \dots \tag{173}$$

The first series in this expression represents the energy of bending of the beam (see eq. 152) and the second series the energy of deformation of the foundation.

The kinetic energy of vibration is, from eq. (153),

$$T = \frac{\gamma A l}{4g}\sum_{i=1}^{i=\infty} \dot{q}_i^2.$$

The differential equation of motion for any coordinate q_i is

$$\ddot{q}_i + \frac{2g}{\gamma A l}\left(\frac{EI\pi^4}{2l^3} i^4 + \frac{kl}{2}\right) q_i = \frac{2g}{\gamma A l} Q_i$$

or

$$\ddot{q}_i + \frac{a^2\pi^4}{l^4}(i^4 + \beta) q_i = \frac{2g}{\gamma A l} Q_i. \tag{b}$$

in which Q_i denotes the external disturbing force corresponding to the coordinate q_i

$$a^2 = \frac{EIg}{\gamma A}; \qquad \beta = \frac{kl^4}{EI\pi^4}. \tag{174}$$

By taking $\beta = 0$, the equation for a hinged bar unsupported by any elastic foundation will be obtained (see p. 348). Denoting,

$$p_i^2 = \frac{a^2\pi^4}{l^4}(i^4 + \beta). \tag{c}$$

a general solution of equation (b) will be

$$q_i = A_i \cos p_i t + B_i \sin p_i t + \frac{2g}{\gamma Al} \cdot \frac{1}{p_i} \int_0^t Q_i \sin p_i(t - t_1)dt_1. \tag{d}$$

The two first terms of this solution represent free vibrations of the beam, depending on the initial conditions. The third term represents vibrations produced by the disturbing force Q_i.

The frequencies of the natural vibrations depend, as seen from (c), not only on the rigidity of the beam but also on the rigidity of the foundation.

As an example consider the case when a pulsating force $P = P_0 \sin \omega t_1$ is acting on the beam at a distance c from the left support (Fig. 182). The generalized force corresponding to the coordinate q_i will be in this case

$$Q_i = P_0 \sin \frac{i\pi c}{l} \sin \omega t_1. \tag{e}$$

Substituting in eq. (d) and considering only vibrations produced by the disturbing force we obtain

$$q_i = \frac{2g}{\gamma A} P_0 \sin \frac{i\pi c}{l} \left\{ \frac{l^3}{\pi^4 a^2 (i^4 + \beta) - \omega^2 l^4} \sin \omega t - \frac{\omega}{lp_i(p_i^2 - \omega^2)} \sin p_i t \right\}.$$

Substituting in (a)

$$y = \frac{2gP_0 l^3}{\gamma A} \sum_{i=1}^{i=\infty} \left\{ \frac{\sin \frac{i\pi c}{l} \sin \frac{i\pi x}{l} \sin \omega t}{\pi^4 a^2 (i^4 + \beta) - \omega^2 l^4} - \frac{\omega \sin \frac{i\pi c}{l} \sin \frac{i\pi x}{l} \sin p_i t}{l^4 p_i (p_i^2 - \omega^2)} \right\}. \tag{f}$$

The first term in this expression represents the forced vibration and the second, the free vibration of the beam. By taking $\omega = 0$ and $P = P_0 \sin \omega t$ the deflection of the beam by a constant force P will be obtained:

$$y = \frac{2Pl^3}{EI\pi^4} \sum_{i=1}^{i=\infty} \frac{\sin \frac{i\pi c}{l} \sin \frac{i\pi x}{l}}{i^4 + \beta}. \tag{175}$$

By taking $c = l/2$ the deflection by the force P at the middle will be obtained as below:

$$y = \frac{2Pl^3}{EI\pi^4} \left(\frac{\sin \frac{\pi x}{l}}{1 + \beta} - \frac{\sin \frac{3\pi x}{l}}{3^4 + \beta} + \frac{\sin \frac{5\pi x}{l}}{5^4 + \beta} - \cdots \right). \tag{176}$$

Comparing this with eq. (h), p. 350, it can be concluded that the additional term β in the denominators represents the effect on the deflection of the beam of the elastic foundation.

By comparing the forced vibrations

$$y = \frac{2gP_0 l^3 \sin \omega t}{\gamma A} \sum_{i=1}^{i=\infty} \frac{\sin \frac{i\pi c}{l} \sin \frac{i\pi x}{l}}{\pi^4 a^2 (i^4 + \beta) - \omega^2 l^4} = \frac{2Pl^3}{EI\pi^4} \sum_{i=1}^{i=\infty} \frac{\sin \frac{i\pi c}{l} \sin \frac{i\pi x}{l}}{i^4 + \beta - \frac{\omega^2 l^4}{\pi^4 a^2}}$$

with the statical deflection (175) it can be concluded that the dynamical deflections can be obtained from the statical formula. It is only necessary to replace β by $\beta - (\omega^2 l^4/\pi^4 a^2)$.

By using the notations (174), we obtain

$$\beta - \frac{\omega^2 l^4}{\pi^4 a^2} = \frac{kl^4}{EI\pi^4} - \frac{\omega^2 l^4 \gamma A}{\pi^4 EIg} = \frac{l^4}{EI\pi^4}\left(k - \frac{\gamma \omega^2 A}{g}\right).$$

This means that the dynamical deflection can be obtained from the statical formula by replacing in it the actual modulus of foundation by a diminished value $k - (\gamma\omega^2 A/g)$ of the same modulus. This conclusion remains true also in the case of an infinitely long bar on an elastic foundation. By using it the deflection of a rail produced by a pulsating load can be calculated.*

62. Ritz Method.†—It has already been shown in several cases in previous chapters (see article 16) that in calculating the frequency of the

* See writer's paper, Statical and Dynamical Stresses in Rails, Intern. Congress for Applied Mechanics, Proceedings, Zürich, 1926, p. 407.

† See Walther Ritz, Gesammelte Werke, p. 265 (1911), Paris.

fundamental type of vibration of a complicated system the approximate method of Rayleigh can be applied. In using this method it is necessary to make some assumption as to the shape of the deflection curve of a vibrating beam or vibrating shaft. The corresponding frequency will then be found from the consideration of the energy of the system. The choosing of a definite shape for the deflection curve in this method is equivalent to introducing some additional constraints which reduces the system to one having a single degree of freedom. Such additional constraints can only increase the rigidity of the system and make the frequency of vibration, as obtained by Rayleigh's method, usually somewhat higher than its exact value. Better approximations in calculating the fundamental frequency and also the frequencies of higher modes of vibration can be obtained by Ritz's method which is a further development of Rayleigh's method.* In using this method the deflection curve representing the mode of vibration is to be taken with several parameters, the magnitudes of which should be chosen in such a manner as to reduce to a minimum the frequency of vibration. The manner of choosing the shape of the deflection curve and the procedure of calculating consecutive frequencies will now be shown for the simple case of the vibration of a uniform string (Fig. 188). Assume that

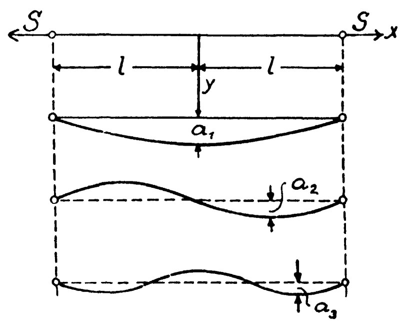

FIG. 188.

S is tensile force in the string,
w is the weight of the string per unit length,
$2l$ is the length of the string.

If the string performs one of the normal modes of vibration, the deflection can be represented as follows:

$$y = X \cos pt, \tag{a}$$

where X is a function of x determining the shape of the vibrating string, and p determines the frequency of vibration. Assuming that the deflec-

* Lord Rayleigh used the method only for an approximate calculation of frequency of the gravest mode of vibration of complicated systems, and was doubtful (see his papers in Phil. Mag., Vol. 47, p. 566; 1899, and Vol. 22, p. 225; 1911) regarding its application to the investigation of higher modes of vibration.

tions are very small, the change in the tensile force S during vibration can be neglected and the increase in potential energy of deformation due to the deflection will be obtained by multiplying S with the increase in length of the string. In this manner the following expression for the potential energy is found, the energy in the position of equilibrium being taken as zero,

$$V = S\int_0^l \left(\frac{dy}{dx}\right)^2 dx.$$

The maximum potential energy occurs when the vibrating string occupies its extreme position. In this position $\cos pt = 1$ and

$$V = S\int_0^l \left(\frac{dX}{dx}\right)^2 dx. \tag{b}$$

The kinetic energy of the vibrating string is

$$T = \frac{w}{g}\int_0^l (\dot{y})^2 dx.$$

Its maximum occurs when the vibrating string is in its middle position, i.e., when $\cos pt = 0$, then

$$T = \frac{p^2 w}{g}\int_0^l X^2 dx. \tag{c}$$

Assuming that there are no losses in energy, we may equate (b) and (c), thus obtaining

$$p^2 = \frac{gS}{w}\,\frac{\int_0^l \left(\frac{dX}{dx}\right)^2 dx}{\int_0^l X^2 dx}. \tag{d}$$

Knowing the shapes of various modes of vibration and substituting in (d) the corresponding expressions for X, the frequencies of these modes of vibration can easily be calculated. In the case of a uniform string, the deflection curves during vibration are sinusoidal curves and for the first three modes of vibration, shown in Fig. 188, we have

$$X_1 = a_1 \cos\frac{\pi x}{2l}; \qquad X_2 = a_2 \sin\frac{\pi x}{l}; \qquad X_3 = a_3 \cos\frac{3\pi x}{2l}.$$

Substituting in (d) we obtain (See eq. 168)

$$p_1{}^2 = \frac{\pi^2}{4l^2}\frac{gS}{w}, \qquad p_2{}^2 = \frac{\pi^2}{l^2}\frac{gS}{w}, \qquad p_3{}^2 = \frac{9}{4}\frac{\pi^2}{l^2}\frac{gS}{w}, \tag{e}$$

and the corresponding frequencies will be

$$f_1 = \frac{p_1}{2\pi} = \frac{1}{4l}\sqrt{\frac{gS}{w}}; \qquad f_2 = \frac{2}{4l}\sqrt{\frac{gS}{w}}; \qquad f_3 = \frac{3}{4l}\sqrt{\frac{gS}{w}}. \tag{f}$$

Let us now apply Ritz's method in calculating from eq. (d) the frequency f_1 of the fundamental type of vibration. The first step in the application of this method is the choosing of a suitable expression for the deflection curve. Let $\varphi_1(x)$, $\varphi_2(x)$, $\cdots$ be a series of functions satisfying the end conditions and suitable for representation of X. Then, by taking

$$X = a_1\varphi_1(x) + a_2\varphi_2(x) + a_3\varphi_3(x) + \cdots, \tag{g}$$

we can obtain a suitable deflection curve of the vibrating string.

We know that by taking a finite number of terms in the expression (g) we superimpose certain limitations on the possible shapes of the deflection curve of the string and due to this fact the frequency, as calculated from (d), will usually be higher than the exact value of this frequency. In order to obtain the approximation as close as possible, Ritz proposed to choose the coefficients a_1, a_2, a_3, $\cdots$ in the expression (g) so as to make the expression (d) a minimum. In this manner a system of equations such as

$$\frac{\partial}{\partial a_n}\frac{\int_0^l \left(\frac{dX}{dx}\right)^2 dx}{\int_0^l X^2 dx} = 0 \tag{h}$$

will be obtained.

Performing the differentiation indicated we have,

$$\int_0^l X^2dx \cdot \frac{\partial}{\partial a_n}\int_0^l \left(\frac{dX}{dx}\right)^2 dx - \int_0^l \left(\frac{dX}{dx}\right)^2 dx \cdot \frac{\partial}{\partial a_n}\int_0^l X^2dx = 0 \tag{k}$$

or noting from (d), that

$$\int_0^l \left(\frac{dX}{dx}\right)^2 dx = \frac{p^2 w}{gS}\int_0^l X^2 dx$$

we obtain, from (k)

$$\frac{\partial}{\partial a_n}\int_0^l \left\{\left(\frac{dX}{dx}\right)^2 - \frac{p^2 w}{gS} X^2\right\} dx = 0. \qquad (l)$$

In this way a system of equations homogeneous and linear in a_1, a_2, a_3, $\cdots$ will be obtained, the number of which will be equal to the number of coefficients a_1, a_2, a_3, $\cdots$ in the expression (g). Such a system of equations can yield for a_1, a_2, a_3 $\cdots$ solutions different from zero only if the determinant of these equations is equal to zero. This condition brings us to the *frequency equation* from which the frequencies of the various modes of vibrations can be calculated.

Let us consider the modes of vibration of a taut string symmetrical with respect to the middle plane. It is easy to see that a function like as $l^2 - x^2$, representing a symmetrical parabolic curve and satisfying end conditions $\{(y)_{x=\pm l} = 0\}$ is a suitable function in this case. By multiplying this function with x^2, x^4, $\cdots$ a series of curves symmetrical and satisfying the end conditions will be obtained. In this manner we arrive at the following expression for the deflection curve of the vibrating string

$$X = a_1(l^2 - x^2) + a_2x^2(l^2 - x^2) + a_3x^4(l^2 - x^2) + \cdots. \qquad (m)$$

In order to show how quickly the accuracy of our calculations increases with an increase in the number of terms of the expression (m) we begin with one term only and put

$$X_1 = a_1(l^2 - x^2).$$

Then,

$$\int_0^l (X_1)^2 dx = \frac{8}{15} a_1^2 l^5; \qquad \int_0^l \left(\frac{dX_1}{dx}\right)^2 dx = \frac{4}{3} a_1^2 l^3.$$

Substituting in eq. (d) we obtain

$$p_1^2 = \frac{5}{2l^2}\frac{gS}{w}.$$

Comparing this with the exact solution (e) it is seen that $5/2$ instead of $\pi^2/4$ is obtained, and the error in frequency is only .66%.

It should be noted that by taking only one term in the expression (m) the shape of the curve is completely determined and the system is reduced to one with a single degree of freedom, as is done in Rayleigh's approximate method.

In order to get a further approximation let us take two terms in the expression (m). Then we will have two parameters a_1 and a_2 and by changing the ratio of these two quantities we can change also, to a certain extent, the shape of the curve. The best approximation will be obtained when this ratio is such that the expression (d) becomes a minimum, which is accomplished when the conditions (l) are satisfied.

By taking

$$X_2 = a_1(l^2 - x^2) + a_2x^2(l^2 - x^2)$$

we obtain

$$\int_0^l X_2{}^2dx = \frac{8}{15}a_1{}^2l^5 + \frac{16}{105}a_1a_2l^7 + \frac{8}{315}a_2{}^2l^9,$$

$$\int_0^l \left(\frac{dX_2}{dx}\right)^2 dx = \frac{4}{3}a_1{}^2l^3 + \frac{8}{15}a_1a_2l^5 + \frac{44}{105}a_2{}^2l^7.$$

Substituting in eq. (l) and taking the derivatives with respect to a_1 and a_2 we obtain

$$\begin{aligned} a_1(1 - 2/5k^2l^2) + a_2l^2(1/5 - 2/35k^2l^2) &= 0, \\ a_1(1 - 2/7k^2l^2) + a_2l^2(11/7 - 2/21k^2l^2) &= 0, \end{aligned} \qquad (n)$$

in which

$$k^2 = \frac{p^2w}{gS}. \qquad (p)$$

The determinant of the equations (n) will vanish when

$$k^4l^4 - 28k^2l^2 + 63 = 0.$$

The two roots of this equation are

$$k_1{}^2l^2 = 2.46744, \qquad k_2{}^2l^2 = 25.6.$$

Remembering that we are considering only modes of vibration symmetrical about the middle and using eq. (p) we obtain for the first and third modes of vibration,

$$p_1{}^2 = \frac{2.46744}{l^2}\frac{gS}{w}, \qquad p_3{}^2 = \frac{25.6}{l^2}\frac{gS}{w}.$$

Comparing this with the exact solutions (e):

$$p_1{}^2 = \frac{\pi^2}{4l^2}\frac{gS}{w} = \frac{2.467401}{l^2}\frac{gS}{w}; \qquad p_3{}^2 = \frac{9}{4}\frac{\pi^2}{l^2}\frac{gS}{w} = \frac{22.207}{l^2}\frac{gS}{w},$$

it can be concluded that the accuracy with which the fundamental frequency is obtained is very high (the error is less than .001%). The error in the frequency of the third mode of vibration is about 6.5%. By taking three terms in the expression (m) the frequency of the third mode of vibration will be obtained with an error less than ½%.*

It is seen that by using the Ritz method not only the fundamental frequency but also frequencies of higher modes of vibration can be obtained with good accuracy by taking a sufficient number of terms in the expression for the deflection curve. In the next article an application of this method to the study of the vibrations of bars of variable cross section will be shown.

63. Vibration of Bars of Variable Cross Section.—*General.*—In our previous discussion various problems involving the vibration of prismatical bars were considered. There exist, however, several important engineering problems such as the vibration of turbine blades, of hulls of ships, of beams of variable depth, etc., in which recourse has to be taken to the theory of vibration of a bar of variable section. The differential equation of vibration of such a bar has been previously discussed (see p. 332) and has the following form,

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right) + \frac{A\gamma}{g}\frac{\partial^2 y}{\partial t^2} = 0, \tag{177}$$

in which I and A are certain functions of x. Only in some special cases which will be considered later, the exact forms of the normal functions can be determined in terms of known functions and usually in the solution of such problems approximate methods like the Rayleigh-Ritz method are used for calculating the natural frequencies of vibration. By taking the deflection of the rod, while vibrating, in the form

$$y = X \cos pt, \tag{a}$$

in which X determines the *mode of vibration*, we obtain the following expressions for the maximum potential and the maximum kinetic energy,

$$V = \frac{1}{2}\int_0^l EI\left(\frac{d^2X}{dx^2}\right)^2 dx, \tag{b}$$

$$T = \frac{p^2}{2g}\int_0^l A\gamma X^2 dx, \tag{c}$$

* See W. Ritz, mentioned above, p. 370.

from which

$$p^2 = \frac{Eg}{\gamma} \frac{\int_0^l I\left(\frac{d^2X}{dx^2}\right)^2 dx}{\int_0^l AX^2dx} \tag{d}$$

The exact solution for the frequency of the fundamental mode of vibration will be the one which makes the left side of (d) a minimum. In order to obtain an approximate solution we proceed as in the previous article and take the shape of the deflection curve in the form of a series,

$$X = a_1\varphi_1(x) + a_2\varphi_2(x) + a_3\varphi_3(x) + \cdots, \tag{e}$$

in which every one of the functions φ satisfies the conditions at the ends of the rod. Substituting (e) in eq. (d) the conditions of minimum will be

$$\frac{\partial}{\partial a_n} \frac{\int_0^l I\left(\frac{d^2X}{dx^2}\right)^2 dx}{\int_0^l AX^2dx} = 0 \tag{f}$$

or

$$\int_0^l AX^2dx\cdot\frac{\partial}{\partial a_n}\int_0^l I\left(\frac{d^2X}{dx^2}\right)^2 dx - \\ - \int_0^l I\left(\frac{d^2X}{dx^2}\right)^2 dx\cdot\frac{\partial}{\partial a_n}\int_0^l AX^2dx = 0. \tag{g}$$

From (g) and (d) we obtain

$$\frac{\partial}{\partial a_n}\int_0^l \left[I\left(\frac{d^2X}{dx^2}\right)^2 - \frac{p^2A\gamma}{Eg}X^2\right]dx = 0. \tag{178}$$

The problem reduces to finding such values for the constants $a_1, a_2, a_3, \cdots$ in eq. (e) as to make the integral

$$S = \int_0^l \left[I\left(\frac{d^2X}{dx^2}\right)^2 - \frac{p^2A\gamma}{Eg}X^2\right]dx \tag{h}$$

a minimum.

The equations (178) are homogeneous and linear in $a_1, a_2, a_3, \cdots$ and their number is equal to the number of terms in the expression (e). Equating to zero the determinant of these equations, the *frequency equation* will be obtained from which the frequencies of the various modes can be calculated.

Vibration of a Wedge.—In the case of a wedge of constant unit thickness with one end free, and the other one built in (Fig. 189) we have

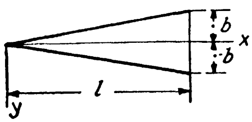

Fig. 189.

$$A = \frac{2bx}{l},$$

$$I = \frac{1}{12}\left(\frac{2bx}{l}\right)^3,$$

where l is the length of the cantilever,

$2b$ is the depth of the cantilever at the built-in end.

The end conditions are:

(1) $\left(EI\dfrac{d^2X}{dx^2}\right)_{x=0} = 0$ (2) $\dfrac{d}{dx}\left(EI\dfrac{d^2X}{dx^2}\right)_{x=0} = 0,$

(3) $(X)_{x=l} = 0,$ (4) $\left(\dfrac{dX}{dx}\right)_{x=l} = 0.$

In order to satisfy the conditions at the ends we take the deflection curve in the form of the series

$$X = a_1\left(1 - \frac{x}{l}\right)^2 + a_2\frac{x}{l}\left(1 - \frac{x}{l}\right)^2 + a_3\frac{x^2}{l^2}\left(1 - \frac{x}{l}\right)^2 + \cdots. \tag{k}$$

It is easy to see that each term as well as its derivative with respect to x, becomes equal to zero when $x = l$. Consequently the end conditions (3) and (4) above will be satisfied. Conditions (1) and (2) are also satisfied since I and dI/dx are zero for $x = 0$.

Taking as a first approximation

$$X_1 = a_1\left(1 - \frac{x}{l}\right)^2$$

and substituting in (d) we obtain

$$p^2 = 10\frac{Eg}{\gamma}\frac{b^2}{l^4}; \qquad f = \frac{p}{2\pi} = \frac{5.48}{2\pi}\frac{b}{l^2}\sqrt{\frac{Eg}{3\gamma}}. \tag{l}$$

In order to get a closer approximation we take two terms in (k), then

$$X_2 = a_1\left(1 - \frac{x}{l}\right)^2 + a_2\frac{x}{l}\left(1 - \frac{x}{l}\right)^2.$$

Substituting in (h)

$$S_2 = \frac{2}{3}\frac{b^3}{l^3}\left((a_1 - 2a_2)^2 + \frac{24}{5}a_2(a_1 - 2a_2) + 6a_2^2\right) - \frac{2b\gamma lp^2}{Eg}\left(\frac{a_1^2}{30} + \frac{2a_1a_2}{105} + \frac{a_2^2}{280}\right).$$

Now from the conditions

$$\frac{\partial S_2}{\partial a_1} = 0, \qquad \frac{\partial S_2}{\partial a_2} = 0,$$

we obtain

$$\left(\frac{Eg}{\gamma}\frac{b^2}{3l^4} - \frac{p^2}{30}\right)a_1 + \left(\frac{2}{5}\frac{Eg}{\gamma}\frac{b^2}{3l^4} - \frac{p^2}{105}\right)a_2 = 0,$$

$$\left(\frac{2}{5}\frac{Eg}{\gamma}\frac{b^2}{3l^4} - \frac{p^2}{105}\right)a_1 + \left(\frac{2}{5}\frac{Eg}{\gamma}\frac{b^2}{3l^4} - \frac{p^2}{280}\right)a_2 = 0.$$

Equating to zero the determinant of these equations we get

$$\left(\frac{Eg}{\gamma}\frac{b^2}{3l^4} - \frac{p^2}{30}\right)\left(\frac{2}{5}\frac{Eg}{\gamma}\frac{b^2}{3l^4} - \frac{p^2}{280}\right) - \left(\frac{2}{5}\frac{Eg}{\gamma}\frac{b^2}{3l^4} - \frac{p^2}{105}\right)^2 = 0. \qquad (m)$$

From this equation p^2 can be calculated. The smallest of the two roots gives

$$f = \frac{p}{2\pi} = \frac{5.319}{2\pi}\frac{b}{l^2}\sqrt{\frac{Eg}{3\gamma}}. \qquad (n)$$

It is interesting to note that for the case under consideration an exact solution exists in which the forms of the normal functions are determined in terms of Bessel's functions.* From this exact solution we have

$$f = \frac{p}{2\pi} = \frac{5.315}{2\pi}\frac{b}{l^2}\sqrt{\frac{Eg}{3\gamma}}. \qquad (179)$$

Comparing with (l) and (n) it can be concluded that the accuracy of the first approximation is about 3%, while the error of the second approximation is less than .1% and a further increase in the number of terms in expression (e) is necessary only if the frequencies of the higher modes of vibration are also to be calculated.

For comparison it is important to note that in the case of a prismatical

* See G. Kirchhoff, Berlin, Monatsberichte, p. 815 (1879), or Ges. Abhandlungen, p. 339. See also Todhunter and Pearson, A History of the Theory of Elasticity, Vol. 2, part 2, p. 92.

cantilever bar having the same section as the wedge at the thick end, the following result was obtained (see p. 344)

$$f = \frac{p}{2\pi} = \frac{a1.875^2}{2\pi l^2} = \frac{3.515b}{2\pi l^2}\sqrt{\frac{Eg}{3\gamma}}.$$

The method developed above can be applied also in cases when A and I are not represented by continuous functions of x. These functions may have several points of discontinuity or may be represented by different mathematical expressions in different intervals along the length l. In such cases the integrals (h) should be subdivided into intervals such that I and A may be represented by continuous functions in each of these intervals. If the functions A and I are obtained either graphically or from numerical tables this method can also be used, it being only necessary to apply one of the approximate methods in calculating the integrals (h). This makes Ritz's method especially suitable in studying the vibration of turbine blades and such structures as bridges and hulls of ships.

Vibration of a Conical Bar.—The problem of the vibrations of a conical bar which has its tip free and the base built in was first treated by Kirchhoff.* For the fundamental mode he obtained in this case

$$f = \frac{p}{2\pi} = \frac{4.359}{2\pi}\frac{r}{l^2}\sqrt{\frac{Eg}{\gamma}}, \tag{180}$$

where r is radius of the base,

l is the length of the bar.

For comparison it should be remembered here that a cylindrical bar of the same length and area of base has the frequency (see above)

$$f = \frac{p}{2\pi} = \frac{a}{2\pi}\frac{1.875^2}{l^2} = \frac{1.758}{2\pi}\frac{r}{l^2}\sqrt{\frac{Eg}{\gamma}}.$$

Thus the frequencies of the fundamental modes of a conical and a cylindrical bars are in the ratio 4.359 : 1.758. The frequencies of the higher modes of vibration of a conical bar can be calculated from the equation

$$f = \frac{p}{2\pi} = \frac{\alpha}{2\pi}\frac{r}{l^2}\sqrt{\frac{Eg}{\gamma}}, \tag{181}$$

in which α has the values given below.†

α_1	α_2	α_3	α_4	α_5	α_6
4.359	10.573	19.225	30.339	43.921	59.956

* Loc. cit., p. 379.

† See Dorothy Wrinch, Proc. Roy. Soc. London, Vol. 101 (1922), p. 493.

Other Cases of Vibration of a Cantilever of Variable Cross Section.—In the general case the frequency of the lateral vibrations of a cantilever can be represented by the equation

$$f = \frac{p}{2\pi} = \frac{\alpha}{2\pi}\frac{i}{l^2}\sqrt{\frac{Eg}{\gamma}}, \tag{182}$$

in which i is radius of gyration of the built-in section,
l is length of the cantilever,
α is constant depending on the shape of the bar and on the mode of vibration.

In the following the values of this constant α for certain particular cases of practical importance are given.

1. If the variations of the cross sectional area and of the moment of inertia, along the axis x, can be expressed in the form,

$$A = ax^m; \qquad I = bx^m, \tag{183}$$

x being measured from the free end, i remains constant along the length of the cantilever and the constant α, in eq. (182) can be represented for the fundamental mode with sufficient accuracy by the equation *

$$\alpha = 3.47(1 + 1.05m).$$

2. If the variation of the cross sectional area and of the moment of inertia along the axis x can be expressed in the form

$$A = a\left(1 - c\frac{x}{l}\right); \qquad I = b\left(1 - c\frac{x}{l}\right), \tag{184}$$

x being measured from the built-in end, then i remains constant along the length of the rod and the quantity α, in eq. (182), will be as given in the table below.†

$c =$ 0	.4	.6	.8	1.0
$\alpha =$ 3.515	4.098	4.585	5.398	7.16

Bar of Variable Cross Section with Free Ends.—Let us consider now the case of a laterally vibrating free-free bar consisting of two equal halves

* See Akimasa Ono, Journal of the Society of Mechanical Engineers, Tokyo, Vol. 27 (1924), p. 467.

† Akimasa Ono, Journal of the Society of Mechanical Engineers, Vol. 28 (1925), p. 429.

joined together at their thick ends (Fig. 190), the left half being generated by revolving the curve

$$y = ax^n \tag{o}$$

about the x axis. The exact solution in terms of Bessel functions has been obtained in this case for certain values of n* and the frequency of the fundamental mode can be represented in the form

$$f = \frac{p}{2\pi} = \frac{\alpha r}{4\pi l^2}\sqrt{\frac{Eg}{\gamma}}, \tag{185}$$

in which r is radius of the thickest cross section,
$2l$ is length of the bar,
α is constant, depending on the shape of the curve (o), the values of which are given in the table below:

$n =$ 0	1/4	1/2	3/4	1
$\alpha =$ 5.593	6.957	8.203	9.300	10.173

The application of integral equations in investigating lateral vibrations of bars of variable cross section has been discussed by E. Schwerin.†

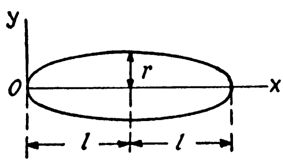

Fig. 190.

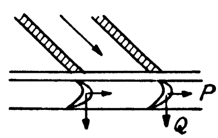

Fig. 191.

64. Vibration of Turbine Blades.—*General.*—It is well known that under certain conditions dangerous vibrations in turbine blades may occur and to this fact the majority of fractures in such blades may be attributed. The disturbing force producing the vibrations in this case is the steam pressure. This pressure always can be resolved into two components; a tangential component P and an axial one Q (Fig. 191) which produce bending of blades in the tangential and axial directions, respectively. These components do not remain constant, but vary with the time because they depend on the relative position of the moving blades with respect to the fixed guide blades. Such pulsating forces, if in resonance with one

* See J. W. Nicholson; Proc. Roy. Soc. of London, Vol. 93 (1917), p. 506.

† E. Schwerin, Über Transversalschwingungen von Stäben veraenderlichen Querschnitts. Zeitschr. f. techn. Physik, Vol. 8, 1927, p. 264.

of the natural modes of vibration of the blades, may produce large forced vibrations with consequent high stresses, which may result finally in the production of progressive fatigue cracks at points of sharp variation in cross section, where high stress concentration takes place. From this it can be seen that the study of vibration of turbine blades and the determination of the various frequencies corresponding to the natural modes of vibration may assist the designer in choosing such proportions for the blades that the possibility of resonance will be eliminated. In making such investigations, Rayleigh's method usually gives a satisfactory approximation. It is therefore unnecessary to go further in the refinement of the calculations, especially if we take into consideration that in actual cases variations in the condition at the built-in end of the blade may affect considerably the frequencies of the natural modes of vibration.*

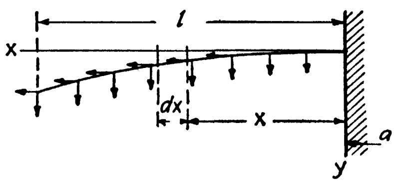

FIG. 192.

Due to the fact that the two principal moments of inertia of a cross section of a blade are different, natural modes of vibration in two principal planes should be studied separately.

Application of Rayleigh's Method.—Let xy be one of these two principal planes (Fig. 192).

l is length of the blade.
a is the radius of the rotor at the built-in end of the blade.
c is constant defined by eq. (184).
A is cross sectional area of the blade varying along the x axis.
ω is angular velocity of the turbine rotor.
γ is weight of material per unit volume.
X is function of x representing the deflection curve of the blade under the action of its weight.

Taking the curve represented by the function X as a basis for the calculation of the fundamental mode of vibration, the deflection curve of the blade during vibration will be,

$$y = X \cos pt. \tag{a}$$

The maximum potential energy will be obtained when the blade is in its extreme position and the deflection curve is represented by the equation

$$y = X. \tag{b}$$

* See W. Hort, V. D. I., Vol. 70 (1926), p. 1420.

This energy consists of two parts: (1) the energy V_1 due to lateral bending and (2) the energy V_2 due to the action of the centrifugal forces. The energy V_1 is equal to the work done by the lateral loading during the deflection, given by eq. (*b*), and is represented by the equation:

$$V_1 = \frac{\gamma}{2}\int_0^l AXdx, \tag{c}$$

in which X, the function of x representing the deflection curve of the blade produced by its weight, can always be obtained by analytical or graphical methods. In the latter case the integral (*c*) can be calculated by one of the approximate methods.

In calculating V_2 it should be noted that the centrifugal force acting on an element of the length dx of the blade (see Fig. 192) is

$$\frac{A\gamma dx}{g}\,\omega^2(a + x). \tag{d}$$

The radial displacement of this element towards the center due to bending of the blade is

$$\frac{1}{2}\int_0^x \left(\frac{dX}{dx}\right)^2 dx, \tag{e}$$

and the work of the centrifugal force (*d*) will be

$$-\frac{A\gamma}{2g}\,dx\omega^2(a + x)\int_0^x \left(\frac{dX}{dx}\right)^2 dx. \tag{f}$$

The potential energy V_2 will now be obtained by the summation of the elements of work (*f*), along the length of the blade and by changing the sign of the sum. Then

$$V_2 = \frac{\gamma\omega^2}{2g}\int_0^l A(a + x)dx\int_0^x \left(\frac{dX}{dx}\right)^2 dx. \tag{g}$$

The maximum kinetic energy will be obtained when the vibrating blade is in its middle position and the velocities, calculated from equation (*a*) have the values:

$$\dot{y} = pX.$$

Then

$$T = \frac{1}{2}\int_0^l \frac{A\gamma}{g}\,\dot{y}^2dx = \frac{\gamma p^2}{2g}\int_0^l AX^2dx. \tag{h}$$

Now, from the equation

$$T = V_1 + V_2$$

we obtain

$$p^2 = \frac{g\int_0^l AXdx + \omega^2\int_0^l A(a + x)dx\int_0^x\left(\frac{dX}{dx}\right)^2 dx}{\int_0^l AX^2dx}. \tag{186}$$

This is the equation for calculating the frequency of the first natural mode of vibration of a blade.

The second term in the numerator of the right-hand member represents the effect of centrifugal force. Denoting by

$$f_1^2 = \frac{g}{(2\pi)^2}\frac{\int_0^l AXdx}{\int_0^l AX^2dx}\,; \quad f_2^2 = \frac{\omega^2}{(2\pi)^2}\frac{\int_0^l A(a + x)dx\int_0^x\left(\frac{dX}{dx}\right)^2 dx}{\int_0^l AX^2dx}. \tag{187}$$

we find, from eq. (186), that the frequency of vibration of the blade can be represented in the following form:

$$f = \sqrt{f_1^2 + f_2^2}, \tag{188}$$

in which f_1 denotes the frequency of the blade when the rotor is stationary, and f_2 represents the frequency of the blade when the elastic forces are neglected and only the restitutive force due to centrifugal action is taken into consideration.

Vibration in the Axial Direction.—In calculating the frequency of vibration in an axial direction a good approximation can be obtained by assuming that the variation of the cross sectional area and of the moment of inertia along the axis of the blade is given by the equations (184). In this case the frequency f_1 will be obtained by using the corresponding table (see p. 381).

The frequency f_2 for the same case, can be easily calculated from eq. (187) and can be represented in the following form

$$f_2 = \frac{\beta\omega}{2\pi}, \tag{189}$$

in which β is a number depending on the proportions of the blade. Several values of β are given in the table below.* Knowing f_1 and f_2 the frequency f will now be obtained from eq. (188).

$a/l=$ \ $c=$	0	.2	.4	.6	.8	1 0
0	1.00	1.00	1.00	1.00	1.00	1.00
1	1.57	1.58	1.59	1.61	1.64	1.71
2	1.98	2.00	2.01	2.04	2.09	2.19
4	2.62	2.64	2.66	2.70	2.77	2.92
6	3.13	3.15	3.18	3.23	3.31	3.50
8	3.56	3.59	3.62	3.68	3.78	4 00
10	3.95	3 98	4.02	4 08	4.19	4 44

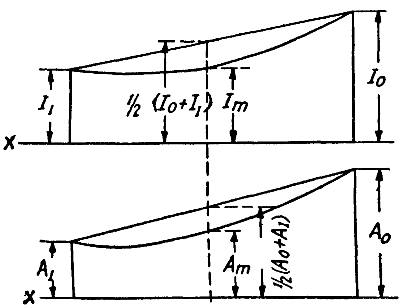

FIG. 193.

Vibration in the Tangential Direction.—In the tangential direction the blades have usually a variable radius of gyration. Consequently the equations (184) cannot be directly applied. In such a case an approximation can be obtained by assuming that the variation of I and A along the x axis (Fig. 193) can be represented by the equations:†

$$I = I_0\left(1 - m\frac{x}{l} - m'\sin\frac{\pi x}{l}\right),$$
$$A = A_0\left(1 - n\frac{x}{l} - n'\sin\frac{\pi x}{l}\right), \tag{190}$$

in which

$$m = \frac{I_0 - I_1}{I_0}; \qquad n = \frac{A_0 - A_1}{A_0};$$

I_m and A_m are the values of I and A at the middle of a blade, and

$$m' = \frac{1}{I_0}\left(\frac{I_0 + I_1}{2} - I_m\right); \qquad n' = \frac{1}{A_0}\left(\frac{A_0 + A_1}{2} - A_m\right).$$

* The table is taken from the paper by Akimasa Ono, mentioned before, p. 381.

† W. Hort: Proceedings of the First International Congress for Applied Mechanics, Delft (1925), p. 282. The numerical results, given below, are obtained on the assumption that the mode of vibration of a bar of variable cross section is the same as that of a prismatical bar.

The frequencies will then be calculated from the general eq. (182) in which the constant α for the fundamental and higher modes of vibration is given by the equation *

$$\alpha_i = \alpha_{0i}\sqrt{\frac{1 - m\beta_i - m'\beta_i'}{1 - n\gamma_i - n'\gamma_i'}}. \tag{191}$$

Here, α_{0i} are values of the constant α for a cantilever of uniform section (see table on p. 344).† The constants β_i, β_i', γ_i and γ_i' for the various modes of vibration are given in the table below.

i	β_i	γ_i	β_i'	γ_i'
1	.193	.807	.493	.493
2	.405	.594	.703	.703
3	.468	.532	.661	.661
4	.483	.517	.649	.649
5	.490	510	.645	645
6	.493	.507	.642	642

If one end of the blade is built in while the other is simply supported, the same equation (191) can be used in calculating α_i. In this case α_{0i} should be taken from the table on p. 345. The constants β_i, β_i', γ_i and γ_i' are given in the table below.

i	β_i	γ_i	β_i'	γ_i'
1	.431	.569	.626	.857
2	.480	.520	.612	.724
3	.490	.510	.623	.680
4	.494	.506	.628	.662
5	.496	.504	.631	.654
6	.497	.503	.633	.649

In this manner f_1 in eq. (188) can be calculated. For calculating f_2 for the fundamental mode, eq. (189) and the above table can be used and the frequency f will then be obtained from eq. (188) as before.

* If the values of m, m', n, and n' are not greater than .5, formula (191) according to Hort, is correct to within 2%. To get an idea of the error made in case m and n were unity, the exact solutions for the natural frequencies of a conical shaped blade and a wedge shaped blade were compared with the values obtained by the above method. It was found that in these extreme cases the error was 17% and 18.5%, respectively, for the conical shaped blade and the wedge shaped blade.

† $k_i^2l^2$ of this table is equal to α_{0i} in eq. (191).

It should be noted that the blades are usually connected in groups by means of shrouding wires. These wires do not always substantially affect the frequencies of the axial vibrations but they may change the frequencies of the tangential vibrations considerably.*

65. Vibration of Hulls of Ships.—As another example of the application of the theory of vibration of bars of variable section, the problem of the vibration of the hull of a ship will now be considered. The disturbing force in this case is usually due to unbalance in the engine or to the action of propellers† and, if the frequency of the disturbing force coincides with the frequency of one of the natural modes of vibration of the hull, large forced vibrations may be produced. If the hull of the ship be taken as a bar of variable section with free ends and Ritz's method (see Art. 62) be applied, the frequencies of the various modes can always be calculated with sufficient accuracy from the eqs. (178).

To simplify the problem let us assume that the bar is symmetrical with respect to the middle cross section and that, by putting the origin of coordinates in this section, the cross sectional area and moment of inertia for any cross section can be represented, respectively, by the equations

$$A = A_0(1 - cx^2); \qquad I = I_0(1 - bx^2), \tag{a}$$

in which A_0 and I_0 denote the cross sectional area and the moment of inertia of the middle cross section, respectively. It is understood that x may vary from $x = -l$ to $x = +l$, $2l$ being the length of the ship.

We will further assume that the deflection during vibration may be represented by

$$y = X \cos pt,$$

in which X is taken in the form of the series,

$$X = a_1\varphi_1(x) + a_2\varphi_2(x) + a_3\varphi_3(x) + \cdots. \tag{b}$$

We must choose for φ_1, φ_2, $\cdots$ suitable functions, satisfying the end conditions. The ratios between the coefficients a_1, a_2, a_3 $\cdots$ and the frequencies will be then obtained from the equations (178).

* See Stodola's book, loc. cit. p. 277. See also W. Hort, V. D. I. Vol. 70 (1926), p. 1422, E. Schwerin, Über die Eigenfrequenzen der Schaufelgruppen von Dampfturbinen, Zeitschr. f. techn. Physik, Vol. 8, 1927, p. 312, and R. P. Kroon, Trans. Am. Soc. Mech. Engrs., V. 56, p. 109, 1934.

† Propeller Vibration is discussed in the paper by F. M. Lewis presented before the "Society of Naval Architects and Marine Engineers," November, 1935, New York.

A satisfactory approximation for the frequency of the fundamental mode of vibration can be obtained * by taking for the functions $\varphi(x)$ the normal functions for a prismatical bar with free ends. The general solution (140) for symmetrical modes of vibration should be taken in the form

$$X = C_1(\cos kx + \cosh kx) + C_2(\cos kx - \cosh kx). \qquad (c)$$

Now from the conditions at the free ends we have

$$(X'')_{x=\pm l} = (X''')_{x=\pm l} = 0. \qquad (d)$$

Substituting (*c*) in (*d*), we obtain

$$\begin{aligned} C_1(-\cos kl + \cosh kl) - C_2(\cos kl + \cosh kl) &= 0, \qquad (e) \\ C_1(\sin kl + \sinh kl) - C_2(-\sin kl + \sinh kl) &= 0. \end{aligned}$$

Putting the determinant of these equations equal to zero the frequency equation

$$\tan kl + \tanh kl = 0, \qquad (f)$$

will be obtained, the consecutive roots of which are

$$k_1l = 0; \qquad k_2l = 2{,}3650 \cdots.$$

Substituting from (*e*) the ratio C_1/C_2 into eq. (*c*) the normal functions corresponding to the fundamental and higher modes of vibration will be

$$X_i = C_i(\cos k_ix \cosh k_il + \cosh k_ix \cos k_il).$$

The arbitrary constant, for simplification, will be taken in the form

$$C_i = \frac{1}{\sqrt{\cos^2 k_il + \cosh^2 k_il}}.$$

The normal function, corresponding to the first root, $k_1l = 0$, will be a constant and the corresponding motion will be a displacement of the bar as a rigid body in the y direction. This constant will be taken equal to $1/\sqrt{2}$.

* See author's book, "Theory of Elasticity," Vol. 2 (1916), S. Petersburg. See also N. Akimoff, Trans. of the Soc. of Naval Arch. (New York), Vol. 26 (1918). Further discussion of the problem is given in the papers by J. Lockwood Taylor, Trans. North East Coast Inst. of Eng. and Shipbuild., 1928 and Trans. of the Instit. of Naval Architects, 1930.

Taking the normal functions, obtained in this manner, as suitable functions $\varphi(x)$ in the series (b) we obtain

$$X = a_1 \frac{1}{\sqrt{2}} + a_2 \frac{\cos k_2 x \cosh k_2 l + \cosh k_2 x \cos k_2 l}{\sqrt{\cos^2 k_2 l + \cosh^2 k_2 l}} + \cdots. \tag{g}$$

Substituting the above in eq. (178), we obtain

$$\begin{aligned} \frac{\partial}{\partial a_n} \Bigg\{ & I_0 \int_{-l}^{+l} (1 - bx^2) \sum_{i=1,2,3} \sum_{j=1,2,3} a_i a_j \varphi_i'' \varphi_j'' dx \\ & - \frac{p^2 A_0 \gamma}{Eg} \int_{-l}^{+l} (1 - cx^2) \sum_{i=1,2,3,\ldots} \sum_{j=1,2,3,\ldots} a_i a_j \varphi_i \varphi_j dx \Bigg\} = 0 \end{aligned} \tag{h}$$

and denoting

$$\int_{-l}^{+l} (1 - bx^2) \varphi_i'' \varphi_j'' dx = \alpha_{ij}; \qquad \int_{-l}^{+l} (1 - cx^2) \varphi_i \varphi_j dx = \beta_{ij} \tag{k}$$

we obtain, from (h),

$$\sum_{i=1,2,3,\cdots} a_i(\alpha_{in} - \lambda\beta_{in}) = 0, \tag{l}$$

in which

$$\lambda = \frac{p^2 A_0 \gamma}{E I_0 g}. \tag{m}$$

For determining the fundamental mode of vibration two terms of the series (g) are practically sufficient. The equations (l) in this case become

$$\begin{aligned} a_1(\alpha_{11} - \lambda\beta_{11}) + a_2(\alpha_{21} - \lambda\beta_{21}) &= 0, \\ a_1(\alpha_{12} - \lambda\beta_{12}) + a_2(\alpha_{22} - \lambda\beta_{22}) &= 0. \end{aligned} \tag{n}$$

In our case,

$$\varphi_1'' = 0; \qquad \varphi_2'' = k_2^2 \frac{-\cos k_2 x \cosh k_2 l + \cosh k_2 x \cos k_2 l}{\sqrt{\cos^2 k_2 l + \cosh^2 k_2 l}}.$$

Substituting this in (k) and performing the integration, we obtain

$$\alpha_{11} = 0; \qquad \alpha_{12} = 0; \qquad \alpha_{21} = 0,$$

$$\alpha_{22} = \int_{-l}^{+l} (1 - bx^2)(\varphi_2'')^2 dx = \frac{31.28}{l^3}(1 - .087bl^2), \tag{p}$$

$$\beta_{11} = l(1 - .333cl^2); \quad \beta_{12} = \beta_{21} = .297cl^3; \quad \beta_{22} = l(1 - .481cl^2). \tag{q}$$

Substituting in eqs. (n) and equating the determinant of these equations to zero, the frequency equation becomes:

$$\lambda^2\left(1 - \frac{\beta_{12}^2}{\beta_{11}\beta_{22}}\right) - \lambda\frac{\alpha_{22}}{\beta_{22}} = 0. \tag{r}$$

The first root of this equation ($\lambda = 0$) corresponds to a displacement of the bar as a rigid body. The second root

$$\lambda = \frac{\alpha_{22}}{\beta_{22}} \frac{1}{1 - \dfrac{\beta_{12}^2}{\beta_{11}\beta_{22}}} \tag{s}$$

determines the frequency of the fundamental type of vibration. This frequency is

$$f_1 = \frac{p}{2\pi} = \frac{\sqrt{\lambda}}{2\pi}\sqrt{\frac{EI_0g}{A_0\gamma}}. \tag{t}$$

Numerical Example. Let $2l = 100$ meters; $J_0 = 20$ (meter)^4; $A_0\gamma = 7 \times 9.81$ ton per meter;* $b = c = .0003$ per meter square. Then the weight of the ship

$$Q = 2A_0\gamma\int_0^l (1 - cx^2)dx = 5150 \text{ ton}.$$

From eqs. (p) and (q) we obtain

$\alpha_{22} = 23.40 \times 10^{-5}$; $\beta_{11} = 37.50$; $\beta_{12} = 11.14$; $\beta_{22} = 31.95$;

then, from eq. (s) we get

$$\lambda = .817 \times 10^{-5}.$$

Assuming $E = 2.10^7$ ton per meter square, we obtain

$$p = \sqrt{\frac{20}{7} \times 2 \times 10^7 \times .817 \times 10^{-5}} = 21.6.$$

The number of oscillations per minute

$$N = \frac{60p}{2\pi} = 206.$$

* To take into account the pulsating current flow in the water due to vibration, certain mass of water must be added to the mass of the hull. This question is discussed in the papers by F. E. Lewis, Proc. Soc. Nav. Archit. and Marine Engrs., New York, November, 1929; E. B. Moulin and A. D. Brown, Proc. Cambridge Phil. Soc., V. 24, pp. 400 and 531, 1928; A. D. Brown, E. B. Moulin and A. J. Perkins, Proc. Cambridge Phil. Soc., V. 26, p. 258, 1930, and J. J. Koch, Ingenieur-Archiv., V. 4, p. 103, 1933.

The functions $\varphi(x)$, taken above, can be used also when the laws of variation of I and A are different from those given by eqs. (a) and also when I and A are given graphically. In each case it is only necessary to calculate the integrals (k) which calculation can always be carried out by means of some approximate method.

66. Lateral Impact of Bars.—*Approximate Solution.*—The problem of stresses and deflections produced in a beam by a falling body is of great practical importance. The exact solution of this problem involves the study of the lateral vibration of the beam. In cases where the mass of the beam is negligible in comparison with the mass of the falling body an approximate solution can easily be obtained by assuming that the deflection curve of the beam during impact has the same shape as the corresponding statical deflection curve. Then the maximum deflection and the maximum stress will be found from a consideration of the energy of the system. Let us take, for example, a beam supported at the ends and struck midway between the supports by a falling weight W. If δ denotes the deflection at the middle of the beam the following relation between the deflection and the force P acting on the beam holds:

$$\delta = \frac{Pl^3}{48EI}$$

and the potential energy of deformation will be

$$V = \frac{P\delta}{2} = \frac{24EI\delta^2}{l^3}. \tag{a}$$

If the weight W falls through a height h, the work done by this load during falling will be

$$W(h + \delta_d) \tag{b}$$

and the dynamical deflection δ_d will be found from the equation,

$$W(h + \delta_d) = \frac{24EI\delta_d{}^2}{l^3}, \tag{c}$$

from which

$$\delta_d = \delta_{st} + \sqrt{\delta_{st}{}^2 + 2h\delta_{st}}, \tag{d}$$

where

$$\delta_{st} = \frac{Wl^3}{48EI}$$

represents the statical deflection of the beam under the action of the load W.

In the above discussion the mass of the beam was neglected and it was assumed that the kinetic energy of the falling weight W was completely transformed into potential energy of deformation of the beam. In actual conditions a part of the kinetic energy will be lost during the impact. Consequently calculations made as above will give an upper limit for the dynamical deflection and the dynamical stresses. In order to obtain a more accurate solution the mass of a beam subjected to impact must be taken into consideration.

If a moving body, having a mass W_1/g and a velocity v_0 strikes centrally a stationary body of mass W_1/g, and, if the deformation at the point of contact is perfectly inelastic, the final velocity v, after the impact (equal for both bodies), may be determined from the equation

$$\frac{W}{g}v_0 = \frac{W + W_1}{g}v,$$

from which

$$v = v_0\frac{W}{W + W_1}. \tag{e}$$

It should be noted that for a beam at the instance of impact, it is only at the point of contact that the velocity v of the body W and of the beam will be the same. Other points of the beam may have velocities different from v, and at the supports of the beam these velocities will be equal to zero. Therefore, not the actual mass of the beam, but some *reduced mass* must be used in eq. (e) for calculating the velocity v. The magnitude of this reduced mass will depend on the shape of the deflection curve and can be approximately determined in the same manner as was done in Rayleigh's method (see eq. 41, p. 85), i.e., by assuming that the deflection curve is the same as the one obtained statically. Then

$$v = v_0\frac{W}{W + \dfrac{17}{35}W_1},$$

in which $17/35W_1$ is the *reduced weight* of the beam. The kinetic energy of the system will be

$$\frac{\left(W + \dfrac{17}{35}W_1\right)v^2}{2g} = \frac{Wv_0^2}{2g}\frac{1}{1 + \dfrac{17}{35}\dfrac{W_1}{W}}.$$

This quantity should be substituted for $(Wv_0^2/2g) = Wh$ in the previous equation (c) in order to take into account the effect of the mass of the beam. The dynamical deflection then becomes

$$\delta_d = \delta_{st} + \sqrt{\delta^2_{st} + 2h\delta_{st}\frac{1}{1 + \frac{17}{35}\frac{W_1}{W}}}. \tag{192}$$

The same method can be used in all other cases of impact in which the displacement of the structure at the point of impact is proportional to the force.*

Impact and Vibrations.—The method described above gives sufficiently accurate results for the cases of thin rods and beams if the mass of the falling weight is large in comparison to the mass of the beam. Otherwise the consideration of vibrations of the beam and of local deformations at the point of impact becomes necessary.

Lateral vibrations of a beam struck by a body moving with a given velocity were considered by S. Venant.† Assuming that after impact the striking body becomes attached to the beam, the vibrations can be investigated by expressing the deflection as the sum of a series of normal functions. The constant coefficients of this series should be determined in such a manner as to satisfy the given initial conditions. In this manner, S. Venant was able to show that the approximate solution given above has an accuracy sufficient for practical applications.

The assumption that after impact the striking body becomes attached to the beam is an aribitrary one and in order to get a more accurate picture of the phenomena of impact, the local deformations of the beam and of the striking body at the point of contact should be investigated. Some results of such an investigation in which a ball strikes the flat surface of a rectangular beam will now be given.‡ The local deformation will be given in this case by the known solution of Herz.§ Let α denote the displacement of the striking ball with respect to the axis of the beam due to this deformation and P, the corresponding pressure of the ball on the beam; then

$$\alpha = kP^{2/3}, \tag{f}$$

* This method was developed by H. Cox, Cambridge Phil. Soc. Trans., Vol. 9 (1850), p. 73. See also Todhunter and Pearson, History, Vol. 1, p. 895.

† Loc. cit., p. 307, *note finale du paragraphe* 61, p. 490.

‡ See author's paper, Zeitschr. f. Math. u. Phys., Vol. 62 (1913), p. 198.

§ H. Herz: J. f. Math. (Crelle), Vol. 92 (1881). A. E. H. Love, Math. Theory of Elasticity (1927), p. 198.

where k is a constant depending on the elastic properties of the bodies and on the magnitude of the radius of the ball. The pressure P, during impact, will vary with the time and will produce a deflection of the beam which can be expressed by the general solution (c) of Art. 58. If the beam is struck at the middle, the expression for the generalized forces will be

$$Q_i = P \sin \frac{i\pi}{2}$$

and the deflection at the middle produced by the pressure P becomes

$$y = \sum_{i=1,3,5,\dots}^{i=\infty} \frac{1}{i^2} \frac{l^2}{\pi^2 a} \frac{2g}{\gamma A l} \int_0^t P \sin \frac{i^2\pi^2 a(t - t_1)dt_1}{l^2}. \tag{g}$$

The complete displacement of the ball from the beginning of the impact ($t = 0$) will be equal to

$$d = \alpha + y. \tag{h}$$

The same displacement can be found now from a consideration of the motion of the ball. If v_0 is the velocity of the ball at the beginning of the impact ($t = 0$) the velocity v at any moment $t = t_1$ will be equal to*

$$v = v_0 - \frac{1}{m} \int_0^{t_1} P dt_1, \tag{k}$$

in which m is the mass of the ball and P is the reaction of the beam on the ball varying with the time. The displacement of the ball in the direction of impact will be,

$$d = v_0 t - \int_0^t \frac{dt_1}{m} \int_0^{t_1} P dt_1. \tag{l}$$

Equating (h) and (l) the following equation is obtained,

$$v_0 t - \int_0^t \frac{dt_1}{m} \int_0^{t_1} P dt_1 = k P^{2/3}$$
$$+ \sum_{i=1,3,5,\dots}^{i=\infty} \frac{1}{i^2} \frac{l^2}{\pi^2 a} \frac{2g}{\gamma A l} \int_0^t P \sin \frac{i^2\pi^2 a(t - t_1)dt_1}{l^2}. \tag{m}$$

This equation can be solved numerically by sub-dividing the interval of time from 0 to t into small elements and calculating, step by step, the

* It is assumed that no forces other than P are acting on the ball.

displacements of the ball. In the following the results of such calculations for two numerical examples are given.

Examples.—In the first example a steel bar of a square cross section 1×1 cm. and of length $= 15.35$ cm. is taken. A steel ball of the radius $r = 1$ cm. strikes the bar with a velocity $v = 1$ cm. per sec. Assuming $E = 2.2 \times 10^6$ kilograms per sq. cm. and $\gamma = 7.96$ grams per cu. cm. the period of the fundamental mode of vibration will be $\tau = .001$ sec. In the numerical solution of eq. (m) this period was sub-divided into 180 equal parts so that $\delta\tau = (1/180)\tau$. The pressure P calculated for each step is given in Fig. 194 by the curve I. For comparison in the same figure the variation of pressure with time, for the case when the ball strikes an infinitely large body having a plane boundary surface is shown by the

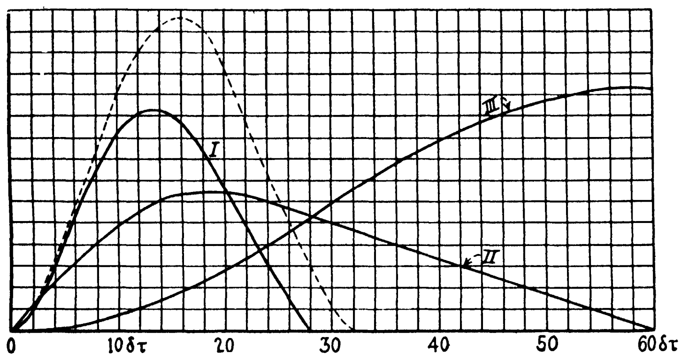

FIG. 194.

dotted lines. It is seen that the ball remains in contact with the bar only during an interval of time equal to 28 ($\delta\tau$), i.e., about 1/6 of τ. The displacements of the ball are represented by curve II and the deflection of the bar at the middle by curve III.

A more complicated case is represented in Fig. 195. In this case the length of the bar and the radius of the ball are taken twice as great as in the previous example. The period τ of the fundamental mode of vibration of the bar is four times as large as in the previous case while the variation of the pressure P is represented by a more complicated curve I. It is seen that the ball remains in contact with the bar from $t = 0$ to $t = 19.5(\delta\tau)$. Then it strikes the bar again at the moment $t = 60(\delta\tau)$ and remains in contact till $t = 80(\delta\tau)$. The deflection of the bar is given by curve II.

It will be noted from these examples that the phenomenon of elastic impact is much more complicated than that of inelastic impact considered by S. Venant.*

67. Longitudinal Impact of Prismatical Bars.—*General.*—For the approximate calculation of the stresses and deflections produced in a

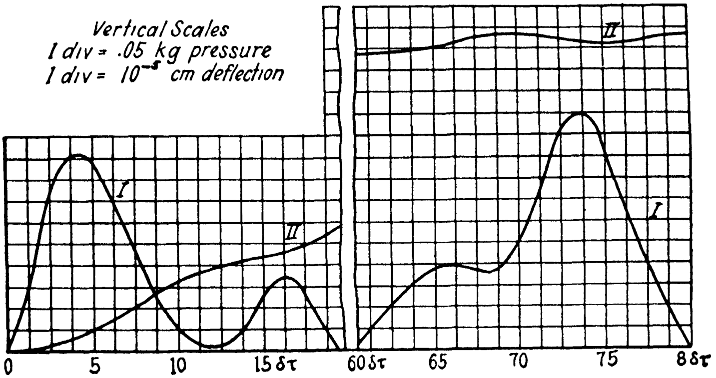

Fig. 195.

prismatical bar, struck longitudinally by a moving body, the approximate method developed in the previous article can be used, but for a more accurate solution of the problem a consideration of the longitudinal vibrations of the bar is necessary.

Young was the first† to point out the necessity of a more detailed consideration of the effect of the mass of the bar on the longitudinal impact. He showed also that any small perfectly rigid body will produce a permanent set in the bar during impact, provided the ratio of the velocity v_1 of motion of the striking body to the velocity v of the propagation of sound waves in the bar is larger than the strain corresponding to the elastic limit in compression of the material. In order to prove this statement he assumed that at the moment of impact (Fig. 196) a local compression will be produced‡ at the surface of contact of the moving body and the bar

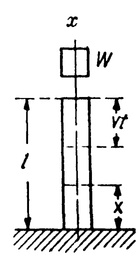

Fig. 196.

* For experimental verification of the above theory see in the paper by H. L. Mason, Trans. Am. Soc. Mech. Engrs., Journal of Applied Mechanics, V. 3, p. 55, 1936.

† See his Lectures on Natural Philosophy, Vol. I, p. 144. The history of the longitudinal impact problem is discussed in detail in the book of Clebsch, translated by S. Venant, loc. cit. p. 307, see *note finale du par.* 60, p. 480, *a*.

‡ It is assumed that the surfaces of contact are two parallel smooth planes.

which compression is propagated along the bar with the velocity of sound. Let us take a very small interval of time equal to t, such that during this interval the velocity of the striking body can be considered as unchanged. Then the displacement of the body will be $v_1 t$ and the length of the compressed portion of the bar will be vt. Consequently the unit compression becomes equal to v_1/v. (Hence the statement mentioned above.)

The longitudinal vibrations of a prismatical bar during impact were considered by Navier.* He based his analysis on the assumption that after impact the moving body becomes attached to the bar at least during a half period of the fundamental type of vibration. In this manner the problem of impact becomes equivalent to that of the vibrations of a load attached to a prismatical bar and having at the initial moment a given velocity (see Art. 52). The solution of this problem, in the form of an infinite series given before, is not suitable for the calculation of the maximum stresses during impact and in the following a more comprehensive solution, developed by S. Venant† and J. Boussinesq,‡ will be discussed.

Bar Fixed at One End and Struck at the Other.§—Considering first the bar fixed at one end and struck longitudinally at the other, Fig. 196, recourse will be taken to the already known equation for longitudinal vibrations (see p. 309). This equation is

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \tag{a}$$

in which u denotes the longitudinal displacements from the position of equilibrium during vibration and

$$a^2 = \frac{Eg}{\gamma}. \tag{b}$$

The condition at the fixed end is

$$(u)_{x=0} = 0. \tag{c}$$

The condition at the free end, at which the force in the bar must be equal to the inertia force of the striking body, will be

$$AE\left(\frac{\partial u}{\partial x}\right)_{x=l} = -\frac{W}{g}\left(\frac{\partial^2 u}{\partial t^2}\right)_{x=l}. \tag{d}$$

* Rapport et Mémoire sur les Ponts Suspendus, Ed. (1823).

† Loc. cit., p. 307.

‡ Applications des Potentiels, p. 508.

§ See Love, "Theory of Elasticity," 4th ed., p. 431 (1927).

Denoting by m the ratio of the weight W of the striking body to the weight $A\gamma l$ of the bar, we obtain, from (d)

$$ml\left(\frac{\partial^2 u}{\partial t^2}\right)_{x=l} = -a^2\left(\frac{\partial u}{\partial x}\right)_{x=l} \tag{e}$$

The conditions at the initial moment $t = 0$, when the body strikes the bar, are

$$u = \frac{\partial u}{\partial t} = 0 \tag{f}$$

for all values of x between $x = 0$ and $x = l$ while at the end $x = l$, since at the instant of impact the velocity of the struck end of the bar becomes equal to that of the striking body, we have:

$$\left(\frac{\partial u}{\partial t}\right)_{t=+0} = -v. \tag{g}$$

The problem consists now in finding such a solution of the equation (a) which satisfies the terminal conditions (c) and (e) and the initial conditions (f) and (g).

The general solution of this equation can be taken in the form

$$u = f(at - x) + f_1(at + x), \tag{h}$$

in which f and f_1 are arbitrary functions.

In order to satisfy the terminal condition (c) we must have,

$$f(at) + f_1(at) = 0$$

or

$$f_1(at) = -f(at) \tag{i}$$

for any value of the argument at. Hence the solution (h) may be written in the form

$$u = f(at - x) - f(at + x). \tag{k}$$

If accents indicate differentiation with respect to the arguments $(at - x)$ or $(at + x)$ and (i) holds we may put

$$\frac{\partial f}{\partial x} = -\frac{\partial f_1}{\partial x} = f'(at - x); \qquad \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f_1}{\partial x^2} = f''(at - x),$$

$$\frac{\partial f}{\partial t} = \frac{\partial f_1}{\partial t} = af'(at - x); \qquad \frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f_1}{\partial t^2} = a^2 f''(at - x);$$

from which it is seen that the expression (k) satisfies eq. (a).

The solution (k) has a very simple physical meaning which can be easily explained in the following manner. Let us take the first term $f(at - x)$ on the right side of eq. (k) and consider a certain instant t. The function f can be represented for this instant by some curve nsr (Fig. 197), the shape of which will depend on the kind of the function f. It is easy to see that after the lapse of an element of time Δt the argument $at - x$ of the function f will remain unchanged provided only that the abscissae are increased during the same interval of time by an element Δx equal to $a\Delta t$. Geometrically this means that during the interval of time Δt the curve nsr moves without distortion to a new position shown in the figure by the dotted line. It can be appreciated from this consideration that the first term on the right side of eq. (k) represents a wave traveling along the x axis with a constant velocity equal to

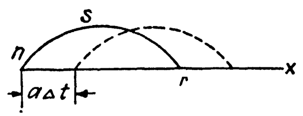

FIG. 197.

$$a = \sqrt{\frac{Eg}{\gamma}}, \tag{193}$$

which is also the velocity of propagation of sound waves along the bar. In the same manner it can be shown that the second term on the right side of eq. (k) represents a wave traveling with the velocity a in the negative direction of the x axis. The general solution (k) is obtained by the superposition of two such waves of the same shape traveling with the same velocity in two opposite directions. The striking body produces during impact a continuous series of such waves, which travel towards the fixed end and are reflected there. The shape of these consecutive waves can now be established by using the initial conditions and the terminal condition at the end $x = l$.

For the initial moment ($t = 0$) we have, from eq. (k),

$$(u)_{t=0} = f(-x) - f(+x),$$

$$\left(\frac{\partial u}{\partial x}\right)_{t=0} = -f'(-x) - f'(+x),$$

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = af'(-x) - af'(+x).$$

Now by using the initial conditions (f) we obtain,

$$\begin{aligned} -f'(-x) - f'(+x) &= 0 \qquad \text{for} \qquad 0 < x < l, \\ f'(-x) - f'(+x) &= 0 \qquad \text{for} \qquad 0 < x < l. \end{aligned} \tag{l}$$

Considering f as a function of an argument z, which can be put equal to $+x$ or $-x$, it can be concluded, from (l), that when $-l < z < l, f'(z)$ is equal to zero, since only under this condition both equations (l) can be satisfied simultaneously and hence $f(z)$ is a constant which can be taken equal to zero and we get,

$$f(z) = 0 \qquad \text{when} \qquad -l < z < l. \tag{m}$$

Now the values of the function $f(z)$ can be determined for the values of z outside the interval $-l < z < l$ by using the end condition (e).

Substituting (k) in eq. (e) we obtain

$$ml\{f''(at - l) - f''(at + l)\} = +f'(at - l) + f'(at + l)$$

or by putting $at + l = z$,

$$f''(z) + \frac{1}{ml}f'(z) = f''(z - 2l) - \frac{1}{ml}f'(z - 2l). \tag{n}$$

By using this equation the function $f(z)$ can be constructed step by step as follows:

From (m) we know that in the interval $l < z < 3l$ the right-hand member of equation (n) is zero. By integrating this equation the function $f(z)$ in the interval $l < z < 3l$ will be obtained. The right-hand member of equation (n) will then become known for the interval $3l < z < 5l$. Consequently the integration of this equation will give the function $f(z)$ for the interval $3l < z < 5l$. By proceeding in this way the function $f(z)$ can be determined for all values of z greater than $-l$.

Considering eq. (n) as an equation to determine $f'(z)$ the general solution of this linear equation of the first order will be

$$f'(z) = Ce^{-z/ml} + e^{-z/ml}\int e^{z/ml}\left(f''(z - 2l) - \frac{1}{ml}f'(z - 2l)\right)dz, \tag{p}$$

in which C is a constant of integration.

For the interval $l < z < 3l$, the right-hand member of eq. (n) vanishes and we obtain

$$f'(z) = Ce^{-z/ml}.$$

Now, by using the condition (g), we have

$$a\{f'(-l + 0) - f'(l + 0)\} = -v$$

or

$$f'(l + 0) = Ce^{-1/m} = \frac{v}{a}; \qquad C = e^{1/m}\frac{v}{a};$$

and we obtain for the interval $l < z < 3l$

$$f'(z) = \frac{v}{a} e^{-(z-l)/ml}. \tag{q}$$

When $3l < z < 5l$, we have, from eq. (q)

$$f'(z - 2l) = \frac{v}{a} e^{-(z-3l)/ml}$$

and

$$f''(z - 2l) - \frac{1}{ml} f'(z - 2l) = -\frac{2}{ml}\frac{v}{a} e^{-(z-3l)/ml}.$$

Now the solution (p) can be represented in the following form,

$$f'(z) = Ce^{-z/ml} - \frac{2}{ml}\frac{v}{a}(z - 3l)e^{-(z-3l)/ml}. \tag{r}$$

The constant of integration C will be determined from the condition of continuity of the velocity at the end $x = l$ at the moment $t = (2l/a)$. This condition is

$$\left(\frac{\partial u}{\partial t}\right)_{\substack{t=2l/a-0\\x=l}} = \left(\frac{\partial u}{\partial t}\right)_{\substack{t=2l/a+0\\x=l}}$$

or by using eq. (k)

$$f'(l - 0) - f'(3l - 0) = f'(l + 0) - f'(3l + 0).$$

Using now eqs. (m) (q) and (r) we obtain

$$-\frac{v}{a} e^{-2/m} = \frac{v}{a} - Ce^{-3/m},$$

from which

$$C = \frac{v}{a}(e^{1/m} + e^{3/m})$$

and we have for the interval $3l < z < 5l$

$$f'(z) = \frac{v}{a} e^{-(z-l)/ml} + \frac{v}{a}\left(1 - \frac{2}{ml}(z - 3l)\right) e^{-(z-3l)/ml}. \tag{s}$$

Knowing $f'(z)$ when $3l < z < 5l$ and using eq. (n), the expression for $f'(z)$ when $5l < z < 7l$ can be obtained and so on.

The function $f(z)$ can be determined by integration if the function $f'(z)$ be known, the constant of integration being determined from the condition that there is no abrupt change in the displacement u at $x = l$. In this manner the following results are obtained when $l < z < 3l$.

$$f(z) = mlv/a\left\{1 - e^{-(z-l)/ml}\right\} \tag{t}$$

when $3l < z < 5l$

$$f(z) = -\frac{mlv}{a}e^{-(z-l)/ml} + \frac{mlv}{a}\left(1 + \frac{2}{ml}(z - 3l)\right)e^{-(z-3l)/ml}. \tag{v}$$

Knowing $f(z)$ the displacements and the stresses at any cross section of the bar can be calculated by substituting in eq. (k) the corresponding values for the functions $f(at - x)$ and $f(at + x)$. When $0 < t < (l/a)$ the term $f(at - x)$ in eq. (k) is equal to zero, by virtue of (m) and hence we have only the wave $f(at + x)$ advancing in the negative direction of the x axis. The shape of this wave will be obtained from (t) by substituting $at + x$ for z. At $t = (l/a)$ this wave will be reflected from the fixed end and in the interval $(l/a) < t < (2l/a)$ we will have two waves, the wave $f(at - x)$ traveling in the positive direction along the x axis and the wave $f(at + x)$ traveling in the negative direction. Both waves can be obtained from (t) by substituting, for z, the arguments $(at - x)$ and $(at + x)$, respectively. Continuing in this way the complete picture of the phenomenon of longitudinal impact can be secured.

The above solution represents the actual conditions only as long as there exists a positive pressure between the striking body and the bar, i.e., as long as the unit elongation

$$\left(\frac{\partial u}{\partial x}\right)_{x=l} = -f'(at - l) - f'(at + l) \tag{w}$$

remains negative. When $0 < at < 2l$, the right-hand member of the eq. (w) is represented by the function (q) with the negative sign and remains negative. When $2l < at < 4l$ the right side of the eq. (w) becomes

$$-\frac{v}{a}e^{-at/ml}\left\{1 + 2e^{2/m}\left(1 - \frac{at - 2l}{ml}\right)\right\}.$$

This vanishes when

$$1 + 2e^{2/m}\left(1 - \frac{at - 2l}{ml}\right) = 0$$

or

$$2at/ml = 4/m + 2 + e^{-2/m}. \tag{x}$$

This equation can have a root in the interval, $2l < at < 4l$ only if

$$2 + e^{-2/m} < 4/m,$$

which happens for $m = 1.73$.

Hence, if the ratio of the weight of the striking body to the weight of the bar is less than 1.73 the impact ceases at an instant in the interval $2l < at < 4l$ and this instant can be calculated from equation (x). For larger values of the ratio m, an investigation of whether or not the impact ceases at some instant in the interval $4l < at < 6l$ should be made, and so on.

The maximum compressive stresses during impact occur at the fixed end and for large values of m ($m > 24$) can be calculated with sufficient accuracy from the following approximate formula:

$$\sigma_{\max} = E\frac{v}{a}(\sqrt{m} + 1). \tag{194}$$

For comparison it is interesting to note that by using the approximate method of the previous article and neglecting δ_{st} in comparison with h in eq. (d) (see p. 392) we arrive at the equation

$$\sigma_{\max} = E\frac{v}{a}\sqrt{m}. \tag{195}$$

When $5 < m < 24$ the equation

$$\sigma_{\max} = E\frac{v}{a}(\sqrt{m} + 1.1) \tag{196}$$

should be used instead of eq. (194). When $m < 5$, S. Venant derived the following formula,

$$\sigma_{\max} = 2E\frac{v}{a}(1 + e^{-2/m}). \tag{197}$$

By using the above method the case of a rod free at one end and struck longitudinally at the other and the case of longitudinal impact of two prismatical bars can be considered.* It should be noted that the investigation of the longitudinal impact given above is based on the assumption that the surfaces of contact between the striking body and the bar are two ideal smooth parallel planes. In actual conditions, there will always be some surface irregularities and a certain interval of time is required to

* See A. E. H. Love, p. 435, loc. cit.

flatten down the high spots. If this interval is of the same order as the time taken for a sound wave to pass along the bar, a satisfactory agreement between the theory and experiment cannot be expected.* Much better results will be obtained if the arrangement is such that the time l/a is comparatively long. For example, by replacing the solid bar by a helical spring C. Ramsauer obtained † a very good agreement between theory and experiment. For this reason we may also expect satisfactory results in applying the theory to the investigation of the propagation of impact waves in long uniformly loaded railway trains. Such a problem may be of practical importance in studying the forces acting in couplings between cars.‡

Another method of obtaining better agreement between theory and experiment is to make the contact conditions more definite. By taking, for instance, a bar with a rounded end and combining the Herz theory for the local deformation at the point of contact with S. Venant's theory of the waves traveling along the bar, J. E. Sears § secured a very good coincidence between theoretical and experimental results.

68. Vibration of a Circular Ring.—The problem of the vibration of a circular ring is encountered in the investigation of the frequencies of vibration of various kinds of circular frames for rotating electrical machinery as is necessary in a study of the causes of noise produced by such machinery. In the following, several simple problems on the vibration of a circular ring of constant cross section are considered, under the assumptions that the cross sectional dimensions of the ring are small in comparison with the radius of its center line and that one of the principal axes of the cross section is situated in the plane of the ring.

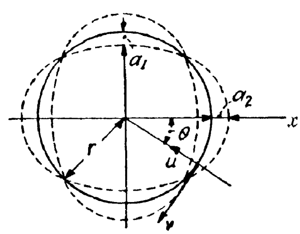

FIG. 198.

Pure Radial Vibration.—In this case the center line of the ring forms a circle of periodically varying radius and all the cross sections move radially without rotation.

* Such experiments with solid steel bars were made by W. Voigt, Wied. Ann., Vol. 19, p. 43 (1883).

† Ann. d. Phys., Vol. 30 (1909).

‡ This question has been studied in the recent paper by O. R. Wikander, Trans. Am. Soc. Mech. Engrs., V. 57, p. 317, 1935.

§ Trans. Cambridge Phil. Soc., Vol. 21 (1908), p. 49. Further experiments are described by J. E. P. Wagstaff, London, Royal Soc. Proc. (ser. A), Vol. 105, p. 544 (1924). See also W. A. Prowse, Phil. Mag., ser. 7, V. 22, p. 209, 1936.

Assume that r is radius of the center line of the ring,
u is the radial displacement, the same for all cross sections.
A is the cross sectional area of the ring.

The unit elongation of the ring in the circumferential direction is then $-u/r$. The potential energy of deformation, consisting in this case of the energy of simple tension will be given by the equation:

$$V = \frac{AEu^2}{2r^2} 2\pi r, \tag{a}$$

while the kinetic energy of vibration will be

$$T = \frac{A\gamma}{2g} \dot{u}^2 2\pi r. \tag{b}$$

From (a) and (b) we obtain

$$\ddot{u} + \frac{Eg}{\gamma} \frac{1}{r^2} u = 0,$$

from which

$$u = C_1 \cos pt + C_2 \sin pt,$$

where

$$p = \sqrt{\frac{Eg}{\gamma r^2}}.$$

The frequency of pure radial vibration is therefore *

$$f = \frac{p}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{Eg}{\gamma r^2}}. \tag{198}$$

A circular ring possesses also modes of vibration analogous to the longitudinal vibrations of prismatical bars. If i denotes the number of wave lengths to the circumference, the frequencies of the higher modes of extensional vibration of the ring will be determined from the equation, †

$$f_i = \frac{1}{2\pi}\sqrt{\frac{Eg}{\gamma r^2}}\sqrt{1 + i^2}. \tag{199}$$

* If there is any additional load, which can be considered as uniformly distributed along the center line of the ring, it is only necessary in the above calculation (eq. b) to replace $A\gamma$ by $A\gamma + w$, where w denotes the additional weight per unit length of the center line of the ring.

† See A. E. H. Love, p. 454, loc. cit.

Torsional Vibration.—Consideration will now be given to the simplest mode of torsional vibration, i.e., that in which the center line of the ring remains undeformed and all the cross sections of the ring rotate during vibration through the same angle (Fig. 199). Due to this rotation a point M, distant y from the middle plane of the ring, will have a radial displacement equal to $y\varphi$ and the corresponding circumferential elongation can be taken approximately equal to $y\varphi/r$. The potential energy of deformation of the ring can now be calculated as follows:

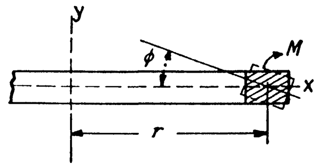

FIG. 199.

$$V = 2\pi r \int_A \frac{E}{2}\left(\frac{y\varphi}{r}\right)^2 dA = \frac{\pi E I_x \varphi^2}{r}, \tag{c}$$

where I_x is moment of inertia of the cross section about the x axis.

The kinetic energy of vibration will be

$$T = 2\pi r \cdot \frac{I_p \gamma}{2g}\dot{\varphi}^2, \tag{d}$$

where I_p is the polar moment of inertia of the cross section.

From (c) and (d) we obtain

$$\ddot{\varphi} + \frac{Eg}{\gamma r^2}\frac{I_x}{I_p}\varphi = 0,$$

from which

$$\varphi = C_1 \cos pt + C_2 \sin pt,$$

where

$$p = \sqrt{\frac{Eg}{\gamma r^2}\frac{I_x}{I_p}}.$$

The frequency of torsional vibration will then be given by

$$f = \frac{1}{2\pi}\sqrt{\frac{Eg}{\gamma r^2}\frac{I_x}{I_p}}. \tag{200}$$

Comparing this result with formula (198) it can be concluded that the frequencies of the torsional and pure radial vibrations are in the ratio $\sqrt{I_x/I_p}$. The frequencies of the higher modes of torsional vibration are given,* in the case of a circular cross section of the ring, by the equation,

$$f_i = \frac{1}{2\pi}\sqrt{\frac{Eg}{2\gamma r^2}}\sqrt{1 + i^2}. \tag{201}$$

* See A. E. H. Love, p. 453, loc. cit.

Remembering that

$$\sqrt{\frac{Eg}{\gamma r^2}} = \frac{a}{r},$$

where a is the velocity of propagation of sound along the bar, it can be concluded that the extensional and torsional vibrations considered above have usually high frequencies. Much lower frequencies will be obtained if flexural vibrations of the ring are considered.

Flexural Vibrations of a Circular Ring.—Flexural vibrations of a circular ring fall into two classes, i.e., flexural vibrations in the plane of the ring and flexural vibrations involving both displacements at right angles to the plane of the ring and twist.* In considering the flexural vibrations in the plane of the ring (Fig. 198) assume that

θ is the angle determining the position of a point on the center line.
u is radial displacement, positive in the direction towards the center.
v is tangential displacement, positive in the direction of the increase in the angle θ.
I is moment of inertia of the cross section with respect to a principal axis at right angles to the plane of the ring.

The unit elongation of the center line at any point, due to the displacements u and v is,

$$e = -\frac{u}{r} + \frac{\partial v}{r\partial\theta} \tag{e}$$

and the change in curvature can be represented by the equation †

$$\frac{1}{r + \Delta r} - \frac{1}{r} = \frac{\partial^2 u}{r^2 \partial\theta^2} + \frac{u}{r^2}. \tag{f}$$

In the most general case of flexural vibration the radial displacement u can be represented in the form of a trigonometrical series ‡

$$u = a_1 \cos\theta + a_2 \cos 2\theta + \cdots + b_1 \sin\theta + b_2 \sin 2\theta + \cdots \tag{h}$$

in which the coefficients $a_1, a_2, \cdots b_1, b_2, \cdots$, varying with the time, represent the generalized coordinates.

* A. E. H. Love, loc. cit., p. 451.

† This equation was established by J. Boussinesq: Comptes Rendus., Vol. 97, p. 843 (1883).

‡ The constant term of the series, corresponding to pure radial vibration, is omitted.

Considering flexural vibrations without extension,* we have, from (*e*),

$$u = \frac{\partial v}{\partial \theta}, \tag{g}$$

from which,†

$$v = a_1 \sin\theta + \tfrac{1}{2}a_2 \sin 2\theta + \cdots - b_1 \cos\theta - \tfrac{1}{2}b_2 \cos 2\theta - \cdots. \tag{k}$$

The bending moment at any cross section of the ring will be

$$M = \frac{EI}{r^2}\left(\frac{\partial^2 u}{\partial \theta^2} + u\right),$$

and hence we obtain for the potential energy of bending

$$V = \frac{EI}{2r^4}\int_0^{2\pi}\left(\frac{\partial^2 u}{\partial \theta^2} + u\right)^2 r\,d\theta,$$

or, by substituting the series (*h*) for u and by using the formulae,

$$\int_0^{2\pi} \cos m\theta \cos n\theta d\theta = 0, \quad \int_0^{2\pi} \sin m\theta \sin n\theta d\theta = 0, \quad \text{when} \quad m \neq n,$$

$$\int_0^{2\pi} \cos m\theta \sin m\theta d\theta = 0, \quad \int_0^{2\pi} \cos^2 m\theta d\theta = \int_0^{2\pi} \sin^2 m\theta d\theta = \pi,$$

we get

$$V = \frac{EI\pi}{2r^3}\sum_{i=1}^{i=\infty}(1 - i^2)^2(a_i^2 + b_i^2). \tag{l}$$

The kinetic energy of the vibrating ring is

$$T = \frac{A\gamma}{2g}\int_0^{2\pi}(\dot{u}^2 + \dot{v}^2) r\,d\theta.$$

By substituting (*h*) and (*k*) for u and v, this becomes

$$T = \frac{\pi r A\gamma}{2g}\sum_{i=1}^{i=\infty}\left(1 + \frac{1}{i^2}\right)(\dot{a}_i^2 + \dot{b}_i^2). \tag{m}$$

It is seen that the expressions (*l*) and (*m*) contain only the squares of

* Discussion of flexural vibrations by taking into account also extension see in the papers by F. W. Waltking, Ingenieur-Archiv., V. 5, p. 429, 1934, and K. Federhofer, Sitzungsberichten der Acad. der Wiss. Wien, Abteilung IIa, V. 145, p. 29, 1936.

† The constant of integration representing a rotation of the ring in its plane as a rigid body, is omitted in the expression (*k*).

the generalized coordinates and of the corresponding velocities; hence these coordinates are the *principal or normal coordinates* and the corresponding vibrations are the principal modes of flexural vibration of the ring. The differential equation for any mode of vibration, from (l) and (m), will be

$$\frac{\pi r A \gamma}{g}\left(1 + \frac{1}{i^2}\right)\ddot{a}_i + \frac{EI\pi}{r^3}(1 - i^2)^2 a_i = 0$$

or

$$\ddot{a}_i + \frac{Eg}{\gamma}\frac{I}{Ar^4}\frac{i^2(1 - i^2)^2}{1 + i^2}a_i = 0.$$

Hence the frequency of any mode of vibration is determined by the equation:

$$f_i = \frac{1}{2\pi}\sqrt{\frac{Eg}{\gamma}\frac{I}{Ar^4}\frac{i^2(1 - i^2)^2}{1 + i^2}}. \tag{202}$$

When $i = 1$, we obtain $f_1 = 0$. In this case $u = a_1 \cos \theta$; $v = a_1 \sin \theta$ and the ring moves as a rigid body, a_1 being the displacement in the negative direction of the x axis Fig. 198. When $i = 2$ the ring performs the fundamental mode of flexural vibration. The extreme positions of the ring during this vibration are shown in Fig. 198 by dotted lines.

In the case of flexural vibrations of a ring of circular cross section involving both displacements at right angles to the plane of the ring and twist the frequencies of the principal modes of vibration can be calculated from the equation*

$$f_i = \frac{1}{2\pi}\sqrt{\frac{Eg}{\gamma}\frac{I}{Ar^4}\frac{i^2(i^2 - 1)^2}{i^2 + 1 + \nu}}, \tag{203}$$

in which ν denotes Poisson's ratio.

Comparing (203) and (202) it can be concluded that even in the lowest mode ($i = 2$) the frequencies of the two classes of flexural vibrations differ but very slightly.†

Incomplete Ring.—When the ring has the form of an incomplete circular arc, the problem of the calculation of the natural frequencies of vibration becomes very complicated.‡ The results so far obtained can

* A. E. H. Love, Mathematical Theory of Elasticity, 4th Ed., Cambridge, 1927, p. 453.

† An experimental investigation of ring vibrations in connection with study of gear noise see in the paper by R. E. Peterson, Trans. Am. Soc. Mech. Engrs., V. 52, p. 1, 1930.

‡ This problem has been discussed by H. Lamb, London Math. Soc. Proc., Vol. 19, p. 365 (1888). See also the paper by F. W. Waltking, loc. cit., p. 409.

be interpreted only for the case where the length of the arc is small in comparison to the radius of curvature. In such cases, these results show that natural frequencies are slightly lower than those of a straight bar of the same material, length, and cross section. Since, in the general case, the exact solution of the problem is extremely complicated, at this date only some approximate values for the lowest natural frequency are available, the Rayleigh-Ritz method * being used in their calculation.

69. Vibration of Membranes.—*General.*—In the following discussion it is assumed that the membrane is a perfectly flexible and infinitely thin lamina of uniform material and thickness. It is further assumed that it is uniformly stretched in all directions by a tension so large that the fluctuation in this tension due to the small deflections during vibration can be neglected. Taking the plane of the membrane coinciding with the xy plane, assume that

v is the displacement of any point of the membrane at right angles to the xy plane during vibration.
S is uniform tension per unit length of the boundary.
w is weight of the membrane per unit area.

The increase in the potential energy of the membrane during deflection will be found in the usual way by multiplying the uniform tension S by the increase in surface area of the membrane. The area of the surface of the membrane in a deflected position will be

$$A = \int\int \sqrt{1 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2}\, dxdy$$

or, observing that the deflections during vibration are very small,

$$A = \int\int \left\{1 + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial y}\right)^2\right\} dxdy.$$

Then the increase in potential energy will be

$$V = \frac{S}{2}\int\int \left\{\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right\} dxdy. \qquad (a)$$

* See J. P. DenHartog, The Lowest Natural Frequency of Circular Arcs, Phil. Mag., Vol. 5 (1928), p. 400; also: Vibration of Frames of Electrical Machines, Trans. A.S.M.E. Applied Mech. Div. 1928. Further discussion of the problem see in the papers by K. Federhofer, Ingenieur-Archiv., V. 4, p. 110, and p. 276, 1933. See also the above mentioned paper by F. W. Waltking.

The kinetic energy of the membrane during vibration is

$$T = \frac{w}{2g} \int \int \dot{v}^2 dx dy. \tag{b}$$

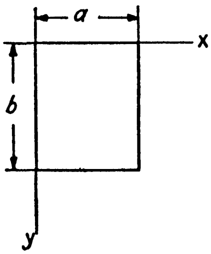

FIG. 200.

By using (*a*) and (*b*) the frequencies of the normal modes of vibration can be calculated as will now be shown for some particular cases.

Vibration of a Rectangular Membrane.—Let a and b denote the lengths of the sides of the membrane and let the axes be taken as shown in Fig. 200. Whatever function of the coordinates v may be, it always can be represented within the limits of the rectangle by the double series

$$v = \sum_{m=1}^{m=\infty} \sum_{n=1}^{n=\infty} q_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \tag{c}$$

the coefficients q_{mn} of which are taken as the generalized coordinates for this case. It is easy to see that each term of the series (*c*) satisfies the boundary conditions, namely, $v = 0$, for $x = 0$; $x = a$ and $v = 0$ for $y = 0$; $y = b$.

Substituting (*c*) in the expression (*a*) for the potential energy we obtain

$$V = \frac{S\pi^2}{2} \int_0^a \int_0^b \left\{ \left(\sum \sum q_{mn} \frac{m}{a} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \right)^2 + \left(\sum \sum q_{mn} \frac{n}{b} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \right)^2 \right\} dx dy.$$

Integrating this expression over the area of the membrane using the formulae of Art. (18) (see p. 99) we find,

$$V = \frac{S}{2} \frac{ab\pi^2}{4} \sum_{m=1}^{m=\infty} \sum_{n=1}^{n=\infty} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) q_{mn}^2. \tag{d}$$

In the same way by substituting (*c*) in eq. (*b*) the following expression for the kinetic energy will be obtained:

$$T = \frac{w}{2g} \frac{ab}{4} \sum \sum \dot{q}_{mn}^2. \tag{e}$$

The expressions (*d*) and (*e*) do not contain the products of the coordinates and of the corresponding velocities, hence the coordinates chosen are principal coordinates and the corresponding vibrations are normal modes of vibration of the membrane.

The differential equation of a normal vibration, from (*d*) and (*e*), will be

$$\frac{w}{g}\frac{ab}{4}\ddot{q}_{mn} + S\frac{ab\pi^2}{4}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)q_{mn} = 0, \tag{f}$$

from which,

$$f_{mn} = \frac{1}{2}\sqrt{\frac{gS}{w}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)}. \tag{204}$$

The lowest mode of vibration will be obtained by putting $m = n = 1$. Then

$$f_{11} = \frac{1}{2}\sqrt{\frac{gS}{w}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)}.$$

The deflection surface of the membrane in this case is

$$v = C\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}. \tag{g}$$

In the same manner the higher modes of vibration can be obtained. Take, for instance, the case of a square membrane, when $a = b$. The frequency of the lowest tone is

$$f_{11} = \frac{1}{a\sqrt{2}}\sqrt{\frac{gS}{w}}. \tag{205}$$

The frequency is directly proportional to the square root of the tension S and inversely proportional to the length of sides of the membrane and to the square root of the load per unit area.

The next two higher modes of vibration will be obtained by taking one of the numbers m, n equal to 2 and the other to 1. These two modes have the same frequency, but show different shapes of deflection surface. In Fig. 201, *a* and *b* the node lines of these two modes of vibration are shown. Because of the fact that the frequencies are the same it is possible to superimpose these two surfaces on each other in any ratio of their maximum deflections. Such a combination is expressed by

$$v = \left(C\sin\frac{2\pi x}{a}\sin\frac{\pi y}{a} + D\sin\frac{\pi x}{a}\sin\frac{2\pi y}{a}\right),$$

where C and D are arbitrary quantities. Four particular cases of such a combined vibration are shown in Fig. 201. Taking $D = 0$ we obtain the vibration mentioned above and shown in Fig. 201, *a*. The membrane, while vibrating, is sub-divided into two equal parts by a vertical nodal

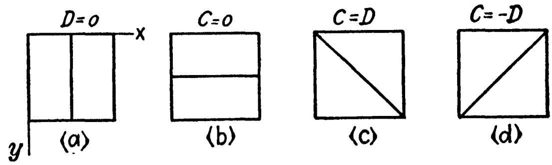

FIG. 201.

line. When $C = 0$, the membrane is sub-divided by a horizontal nodal line as in Fig. 201, *b*. When $C = D$, we obtain

$$v = C\left(\sin\frac{2\pi x}{a}\sin\frac{\pi y}{a} + \sin\frac{\pi x}{a}\sin\frac{2\pi y}{a}\right)$$

$$= 2C\sin\frac{\pi x}{a}\sin\frac{\pi y}{a}\left(\cos\frac{\pi x}{a} + \cos\frac{\pi y}{a}\right).$$

This expression vanishes when

$$\sin\frac{\pi x}{a} = 0, \qquad \text{or} \qquad \sin\frac{\pi y}{a} = 0$$

or again when

$$\cos\frac{\pi x}{a} + \cos\frac{\pi y}{a} = 0.$$

The first two equations give us the sides of the boundary; from the third equation we obtain

$$\frac{\pi x}{a} = \pi - \frac{\pi y}{a}$$

or

$$x + y = a.$$

This represents one diagonal of the square shown in Fig. 201, *d*. Fig. 201, *c* represents the case when $C = -D$. Each half of the membrane in the last two cases can be considered as an isosceles right-angled triangular membrane. The fundamental frequency of this membrane, from eq. (204), will be

$$f = \frac{1}{2}\sqrt{\frac{gS}{w}\left(\frac{4}{a^2} + \frac{1}{a^2}\right)} = \frac{\sqrt{5}}{2a}\sqrt{\frac{gS}{w}}. \tag{206}$$

In this manner also higher modes of vibration of a square or rectangular membrane can be considered.*

In the case of forced vibration of the membrane the differential equation of motion (f) becomes

$$\frac{w}{g}\frac{ab}{4}\ddot{q}_{mn} + S\frac{ab\pi^2}{4}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)q_{mn} = Q_{mn}, \tag{h}$$

in which Q_{mn} is the *generalized disturbing force* corresponding to the coordinate q_{mn}.

Let us consider, as an example, the case of a harmonic force $P = P_0 \cos \omega t$, acting at the center of the membrane. By giving an increase δq_{mn} to a coordinate q_{mn}, in the expression (c), we find for the work done by the force P:

$$P_0 \cos \omega t \delta q_{mn} \sin \frac{m\pi}{2} \sin \frac{n\pi}{2},$$

from which we see that when m and n are both odd, $Q_{mn} = \pm P_0 \cos \omega t$, otherwise $Q_{mn} = 0$. Substituting in eq. (h), and using eq. 48, (p. 104), we obtain

$$q_{mn} = \pm \frac{4g}{abw}\frac{P_0}{p_{mn}}\int_0^t \sin p_{mn}(t - t_1) \cos \omega t_1 dt_1$$

$$= \pm \frac{4g}{abw}\frac{P_0}{p_{mn}^2 - \omega^2}(\cos \omega t - \cos p_{mn}t), \tag{k}$$

where

$$p_{mn}^2 = \frac{gS\pi^2}{w}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

By substituting (k) in the expression (c) the vibrations produced by the disturbing force $P_0 \cos \omega t$ will be obtained.

When a distributed disturbing force of an intensity Z is acting on the membrane, the generalized force in eq. (h) becomes

$$Q_{mn} = \int_0^b \int_0^a Z \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dxdy. \tag{l}$$

* A more detailed discussion of this problem can be found in Rayleigh's book, loc. cit., p. 306. See also Lamé's, Leçons sur l'élasticité. Paris, 1852.

Assume, for instance, that a uniformly distributed pressure Z is suddenly applied to the membrane at the initial moment ($t = 0$), then from (l),

$$Q_{mn} = Z \frac{ab}{mn\pi^2}(1 - \cos m\pi)(1 - \cos n\pi).$$

When m and n both are odd, we have

$$Q_{mn} = \frac{4ab}{mn\pi^2} Z; \tag{m}$$

otherwise Q_{mn} vanishes.

Substituting (m) in eq. (h) and assuming the initial condition that $q_{mn} = 0$ at $t = 0$, we obtain

$$q_{mn} = \frac{16g}{wmn\pi^2} \frac{Z(1 - \cos p_{mn}t)}{p_{mn}^2}. \tag{n}$$

Hence the vibrations produced by the suddenly applied pressure Z are

$$v = \frac{16gZ}{\pi^2 w} \sum\sum \frac{1 - \cos p_{mn}t}{mnp_{mn}^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \tag{o}$$

where m and n are both odd.

Rayleigh-Ritz Method.—In calculating the frequencies of the natural modes of vibration of a membrane the Rayleigh-Ritz method is very useful. In applying this method we assume that the deflections of the membrane, while vibrating, are given by

$$v = v_0 \cos pt, \tag{p}$$

where v_0 is a suitable function of the coordinates x and y which determines the shape of the deflected membrane, i.e., the mode of vibration. Substituting (p) in the expression (a) for the potential energy, we find

$$V_{\max} = \frac{S}{2} \int \int \left\{ \left(\frac{\partial v_0}{\partial x} \right)^2 + \left(\frac{\partial v_0}{\partial y} \right)^2 \right\} dxdy. \tag{q}$$

For the maximum kinetic energy we obtain from (b)

$$T_{\max} = \frac{w}{2g} p^2 \int \int v_0^2 dxdy. \tag{r}$$

Putting (q) equal to (r) we get

$$p^2 = \frac{Sg}{w} \frac{\int \int \left\{ \left(\frac{\partial v_0}{\partial x} \right)^2 + \left(\frac{\partial v_0}{\partial y} \right)^2 \right\} dxdy}{\int \int v_0^2 dxdy}. \tag{s}$$

In applying the Rayleigh-Ritz method we take the expression v_0 for the deflection surface of the membrane in the form of a series:

$$v_0 = a_1\varphi_1(x, y) + a_2\varphi_2(x, y) + a_3\varphi_3(x, y) + \cdots, \tag{t}$$

each term of which satisfies the conditions at the boundary. (The deflections at the boundary of the membrane must be equal to zero.) The coefficients a_1, a_2 $\cdots$ in this series should be chosen in such a manner as to make (s) a minimum, i.e., so as to satisfy all equations of the following form

$$\frac{\partial}{\partial a_n}\frac{\int\int\left\{\left(\frac{\partial v_0}{\partial x}\right)^2+\left(\frac{\partial v_0}{\partial y}\right)^2\right\}dxdy}{\int\int v_0{}^2dxdy} = 0,$$

or

$$\int\int v_0{}^2dxdy\cdot\frac{\partial}{\partial a_n}\int\int\left\{\left(\frac{\partial v_0}{\partial x}\right)^2+\left(\frac{\partial v_0}{\partial y}\right)^2\right\}dxdy -$$
$$-\int\int\left\{\left(\frac{\partial v_0}{\partial x}\right)^2+\left(\frac{\partial v_0}{\partial y}\right)^2\right\}dxdy\cdot\frac{\partial}{\partial a_n}\int\int v_0{}^2dxdy = 0.$$

By using (s) this latter equation becomes

$$\frac{\partial}{\partial a_n}\int\int\left\{\left(\frac{\partial v_0}{\partial x}\right)^2+\left(\frac{\partial v_0}{\partial y}\right)^2-\frac{p^2w}{gS}v_0{}^2\right\}dxdy = 0. \tag{u}$$

In this manner we obtain as many equations of the type (u) as there are coefficients in the series (t). All these equations will be linear in a_1, a_2, a_3, $\cdots$, and by equating the determinant of these equations to zero the frequency equation for the membrane will be obtained.

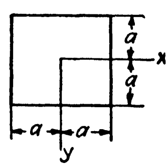

FIG. 202.

Considering, for instance, the modes of vibration of a square membrane symmetrical with respect to the x and y axes, Fig. 202, the series (t) can be taken in the following form,

$$v_0 = (a^2 - x^2)(a^2 - y^2)\,(a_1 + a_2x^2 + a_3y^2 + a_4x^2y^2 + \cdots).$$

It is easy to see that each term of this series becomes equal to zero, when $x = y = \pm\, a$. Hence the conditions at the boundary are satisfied.

In the case of a convex polygon the boundary conditions will be satisfied by taking

$$v_0 = (a_1x + b_1y + c_1)\,(a_2x + b_2y + c_2)\cdots(a_nx + b_ny + c_n)\sum\sum a_{mn}x^my^n,$$

where $a_1x + b_1y + c_1 = 0, \cdots$ are the equations of the sides of the polygon. By taking only the first term ($m = 0$, $n = 0$) of this series a satisfactory approximation for the fundamental type of vibration usually will be obtained. It is necessary to take more terms if the frequencies of higher modes of vibration are required.

Circular Membrane.—We will consider the simplest case of vibration, where the deflected surface of the membrane is symmetrical with respect to the center of the circle. In this case the deflections depend only on the radial distance r and the boundary condition will be satisfied by taking

$$v_0 = a_1 \cos\frac{\pi r}{2a} + a_2 \cos\frac{3\pi r}{2a} + \cdots, \tag{v}$$

where a denotes the radius of the boundary.

Because we are using polar coordinates, eq. (q) has to be replaced in this case by the following equation:

$$V_{\max} = \frac{S}{2}\int_0^a \left(\frac{\partial v_0}{\partial r}\right)^2 2\pi r dr. \tag{q)'$$

Instead of (r) we obtain

$$T_{\max} = \frac{w}{2g}p^2\int_0^a v_0^2 2\pi r dr \tag{r)'$$

and eq. (u) assumes the form

$$\frac{\partial}{\partial a_n}\int_0^a \left\{\left(\frac{\partial v_0}{\partial r}\right)^2 - \frac{p^2 w}{gS} v_0^2\right\} 2\pi r dr = 0. \tag{u)'$$

By taking only the first term in the series (v) and substituting $v_0 = a_1 \cos \pi r/2a$ in eq. (u)' we obtain

$$\frac{\pi^2}{4a^2}\int_0^a \sin^2\frac{\pi r}{2a} r dr = \frac{p^2 w}{gS}\int_0^a \cos^2\frac{\pi r}{2a} r dr,$$

from which

$$\frac{\pi^2}{4a^2}\left(\frac{1}{2} + \frac{2}{\pi^2}\right) = \frac{p^2 w}{gS}\left(\frac{1}{2} - \frac{2}{\pi^2}\right)$$

or

$$p = \frac{2.415}{a}\sqrt{\frac{gS}{w}}.$$

The exact solution * gives for this case,

$$p = \frac{2.404}{a}\sqrt{\frac{gS}{w}}. \tag{207}$$

The error of the first approximation is less than ½%.

In order to get a better approximation for the fundamental note and also for the frequencies of the higher modes of vibration, a larger number of terms in the series (v) should be taken. These higher modes of vibration will have one, two, three, $\cdots$ nodal circles at which the displacements v are zero during vibration.

In addition to the modes of vibration symmetrical with respect to the center a circular membrane may have also modes in which one, two, three,

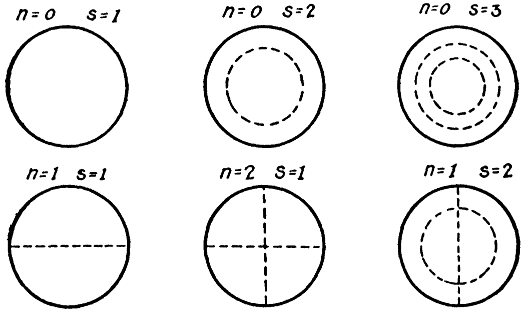

FIG. 203.

$\cdots$ diameters of the circle are *nodal lines*, along which the deflections during vibration are zero. Several modes of vibration of a circular membrane are shown in Fig. 203 where the nodal circles and nodal diameters are indicated by dotted lines.

In all cases the quantity p, determining the frequencies, can be expressed by the equation,

$$p_{ns} = \frac{\alpha_{ns}}{a}\sqrt{\frac{gS}{w}}. \tag{208}$$

the constants α_{ns} of which are given in the table below.† In this table n denotes the number of nodal diameters and s the number of nodal circles. (The boundary circle is included in this number.)

* The problem of the vibration of a circular membrane is discussed in detail by Lord Rayleigh, loc. cit., p. 318.

† The table was calculated by Bourget, Ann. de. l'école normale, Vol. 3 (1866).

s	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
1	2.404	3.832	5 135	6.379	7 586	8.780
2	5.520	7.016	8.417	9.760	11.064	12.339
3	8.654	10.173	11.620	13.017	14.373	15.700
4	11.792	13.323	14.796	16.224	17.616	18.982
5	14 931	16 470	17.960	19 410	20 827	22.220
6	18.071	19.616	21.117	22.583	24 018	25.431
7	21.212	22.760	24.270	25.749	27.200	28 628
8	24.353	25.903	27.421	28.909	30.371	31.813

It is assumed in the previous discussion that the membrane has a complete circular area and that it is fixed only on the circular boundary, but it is easy to see that the results obtained include also the solution of other problems such as membranes bounded by two concentric circles and two radii or membranes in the form of a sector. Take, for instance, membrane semi-circular in form. All possible modes of vibration of this membrane will be included in the modes which the circular membrane may perform. It is only necessary to consider one of the nodal diameters of the circular membrane as a fixed boundary. When the boundary of a membrane is approximately circular, the lowest tone of such a membrane is nearly the same as that of circular membrane having the same area and the same value of Sg/w. Taking the equation determining the frequency of the fundamental mode of vibration of a membrane in the form,

$$p = \alpha\sqrt{\frac{gS}{wA}}, \tag{209}$$

where A is the area of the membrane, the constant α of this equation will be given by the table on page 421, which shows the effect of a greater or less departure from the circular form.*

In cases where the boundary is different from those discussed above the investigation of the vibrations presents mathematical difficulties and only the case of an elliptical boundary has been completely solved by Mathieu.† A complete discussion of the theory of vibration of membranes from a mathematical point of view is given in a book by Pockels.‡

* The table is taken from Rayleigh's book, loc. cit., p. 345.

† Journal de Math. (Liouville), Vol. 13 (1868).

‡ Pockels: Über die partielle Differentialgleichung, $\Delta u + k^2 u = 0$; Leipzig, 1891.

Circle	$\alpha = 2.404\sqrt{\pi}$	$= 4.261$
Square	$\alpha = \pi\sqrt{2}$	$= 4.443$
Quadrant of a circle	$\alpha = \frac{5\ 135}{2}\sqrt{\pi}$	$= 4.551$
Sector of a circle 60°	$\alpha = 6\ 379\sqrt{\frac{\pi}{6}}$	$= 4.616$
Rectangle 3 × 2	$\alpha = \sqrt{\frac{13}{6}} \cdot \pi$	$= 4.624$
Equilateral triangle	$\alpha = 2\pi\sqrt{\tan 30°}$	$= 4.774$
Semi-circle	$\alpha = 3.832\sqrt{\frac{\pi}{2}}$	$= 4.803$
Rectangle 2 × 1	$\alpha = \pi\sqrt{\frac{5}{2}}$	$= 4.967$
Rectangle 3 × 1	$\alpha = \pi\sqrt{\frac{10}{3}}$	$= 5.736$

70. Vibration of Plates.—*General.*—In the following discussion it is assumed that the plate consists of a perfectly elastic, homogeneous, isotropic material and that it has a uniform thickness h considered small in comparison with its other dimensions. We take for the xy plane the middle plane of the plate and assume that with small deflections * the lateral sides of an element, cut out from the plate by planes parallel to the zx and zy planes (see Fig. 204) remain plane and rotate so as to be normal to the deflected middle surface of the plate. Then the strain in a thin layer of this element, indicated by the shaded area and distant z from the middle plane can be obtained from a simple geometrical consideration and will be represented by the following equations:†

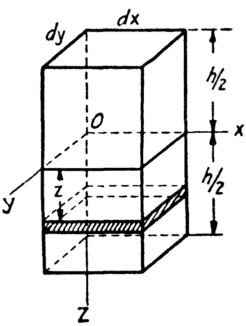

Fig. 204.

$$e_{xx} = \frac{z}{R_1} = -z\frac{\partial^2 v}{\partial x^2},$$

* The deflections are assumed to be small in comparison with the thickness of the plate.

† It is assumed that there is no stretching of the middle plane.

$$e_{yy} = \frac{z}{R_2} = -z\frac{\partial^2 v}{\partial y^2}, \qquad (a)$$
$$e_{xy} = -2z\frac{\partial^2 v}{\partial x \partial y},$$

in which

e_{xx}, e_{yy} are unit elongations in the x and y directions,
e_{xy} is shear deformation in the xy plane,
v is deflection of the plate,
$\frac{1}{R_1}$, $\frac{1}{R_2}$ are curvatures in the xz and yz planes,
h is thickness of the plate.

The corresponding stresses will then be obtained from the known equations:

$$\sigma_x = \frac{E}{1-\nu^2}(e_{xx} + \nu e_{yy}) = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 v}{\partial x^2} + \nu\frac{\partial^2 v}{\partial y^2}\right),$$
$$\sigma_y = \frac{E}{1-\nu^2}(e_{yy} + \nu e_{xx}) = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 v}{\partial y^2} + \nu\frac{\partial^2 v}{\partial x^2}\right), \qquad (b)$$
$$\tau = Ge_{xy} = -\frac{Ez}{(1+\nu)}\cdot\frac{\partial^2 v}{\partial x \partial y},$$

in which ν denotes Poisson's ratio.

The potential energy accumulated in the shaded layer of the element during the deformation will be

$$dV = \left(\frac{e_{xx}\sigma_x}{2} + \frac{e_{yy}\sigma_y}{2} + \frac{e_{xy}\tau}{2}\right)dxdydz$$

or by using the eqs. (a) and (b)

$$dV = \frac{Ez^2}{2(1-\nu^2)}\left\{\left(\frac{\partial^2 v}{\partial x^2}\right)^2 + \left(\frac{\partial^2 v}{\partial y^2}\right)^2 + 2\nu\frac{\partial^2 v}{\partial x^2}\frac{\partial^2 v}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2 v}{\partial x \partial y}\right)^2\right\}dxdydz, \qquad (c)$$

from which, by integration, we obtain the potential energy of bending of the plate

$$V = \int\int\int dV = \frac{D}{2}\int\int\left\{\left(\frac{\partial^2 v}{\partial x^2}\right)^2 + \left(\frac{\partial^2 v}{\partial y^2}\right)^2 + 2\nu\frac{\partial^2 v}{\partial x^2}\frac{\partial^2 v}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2 v}{\partial x \partial y}\right)^2\right\}dxdy, \qquad (210)$$

where $D = \dfrac{Eh^3}{12(1 - \nu^2)}$ is the *flexural rigidity* of the plate.

The kinetic energy of a vibrating plate will be

$$T = \frac{\gamma h}{2g} \int \int v^2 dx dy, \tag{211}$$

where $\gamma h/g$ is the mass per unit area of the plate.

From these expressions for V and T, the differential equation of vibration of the plate can be obtained.

Vibration of a Rectangular Plate.—In the case of a rectangular plate Fig. (200) with simply supported edges we can proceed as in the case of a rectangular membrane and take the deflection of the plate during vibration in the form of a double series

$$v = \sum_{m=1}^{m=\infty} \sum_{n=1}^{n=\infty} q_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}. \tag{d}$$

It is easy to see that each term of this series satisfies the conditions at the edges, which require that w, $\partial^2 w/\partial x^2$ and $\partial^2 w/\partial y^2$ must be equal to zero at the boundary.

Substituting (d) in eq. 210 the following expression for the potential energy will be obtained

$$V = \frac{\pi^4 ab}{8} D \sum_{m=1}^{m=\infty} \sum_{n=1}^{n=\infty} q_{mn}{}^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2. \tag{212}$$

The kinetic energy will be

$$T = \frac{\gamma h}{2g} \frac{ab}{4} \sum \sum \dot{q}_{mn}{}^2. \tag{213}$$

It will be noted that the expressions (212) and (213) contain only the squares of the quantities q_{mn} and of the corresponding velocities, from which it can be concluded that these quantities are normal coordinates for the case under consideration. The differential equation of a normal vibration will be

$$\frac{\gamma h}{g} \ddot{q}_{mn} + \pi^4 D q_{mn} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 = 0,$$

from which

$$q_{mn} = C_1 \cos pt + C_2 \sin pt,$$

where

$$p = \pi^2 \sqrt{\frac{gD}{\gamma h}} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right). \tag{214}$$

From this the frequencies of the lowest mode and of the higher modes of vibration can be easily calculated. Taking, for instance, a square plate we obtain for the lowest mode of vibration

$$f_1 = \frac{p_1}{2\pi} = \frac{\pi}{a^2}\sqrt{\frac{gD}{\gamma h}}. \tag{215}$$

In considering higher modes of vibration and their nodal lines, the discussion previously given for the vibration of a rectangular membrane can be used. Also the case of forced vibrations of a rectangular plate with simply supported edges can be solved without any difficulty. It should be noted that the cases of vibration of a rectangular plate, of which two opposite edges are supported while the other two edges are free or clamped, can also be solved without great mathematical difficulty.*

The problems of the vibration of a rectangular plate, of which all the edges are free or clamped, are, however, much more complicated. For the solution of these problems, Ritz' method has been found to be very useful.† In using this method we assume

$$v = v_0 \cos pt, \tag{e}$$

where v_0 is a function of x and y which determines the mode of vibration. Substituting (e) in the equations (210) and (211), the following expressions for the maximum potential and kinetic energy of vibration will be obtained:

$$V_{\max} = \frac{D}{2}\int\int\left\{\left(\frac{\partial^2 v_0}{\partial x^2}\right)^2 + \left(\frac{\partial^2 v_0}{\partial y^2}\right)^2 + 2\nu\frac{\partial^2 v_0}{\partial x^2}\frac{\partial^2 v_0}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2 v_0}{\partial x \partial y}\right)^2\right\}dxdy$$

$$T_{\max} = \frac{\gamma h}{2g}p^2\int\int v_0^2 dxdy,$$

from which

$$p^2 = \frac{2g}{\gamma h}\frac{V_{\max}}{\int\int v_0^2 dxdy}. \tag{216}$$

Now we take the function v_0 in the form of a series

$$v_0 = a_1\varphi_1(x, y) + a_2\varphi_2(x, y) + \cdots, \tag{f}$$

* See Voigt, Göttinger Nachrichten, 1893, p. 225.

† See W. Ritz, Annalen der Physik, Vol. 28 (1909), p. 737. See also "Gesammelte Werke" (1911), p. 265.

where $\varphi_1, \varphi_2, \cdots$ are suitable functions of x and y, satisfying the conditions at the boundary of the plate. It is then only necessary to determine the coefficients $a_1, a_2, \cdots$ in such a manner as to make the right member of (216) a minimum. In this way we arrive at a system of equations of the type:

$$\frac{\partial}{\partial a_n}\int\int\left\{\left(\frac{\partial^2 v_0}{\partial x^2}\right)^2 + \left(\frac{\partial^2 v_0}{\partial y^2}\right)^2 + 2\nu\frac{\partial^2 v_0}{\partial x^2}\frac{\partial^2 v_0}{\partial y^2}\right.$$
$$\left. + 2(1-\nu)\left(\frac{\partial^2 v_0}{\partial x\partial y}\right)^2 - \frac{p^2\gamma h}{gD}v_0{}^2\right\} dxdy = 0. \qquad (217)$$

which will be linear with respect to the constants $a_1, a_2, \cdots$ and by equating to zero the determinant of these equations the frequencies of the various modes of vibration can be approximately calculated.

W. Ritz applied this method to the study of the vibration of a square plate with free edges.* The series (f) was taken in this case in the form,

$$v_0 = \sum\sum a_{mn}u_m(x)v_n(y), \qquad (f)'$$

where $u_m(x)$ and $v_n(y)$ are the normal functions of the vibration of a prismatical bar with free ends (see p. 343). The frequencies of the lowest and of the higher modes of vibration will be determined by the equation

$$p = \frac{\alpha}{a^2}\sqrt{\frac{gD}{\gamma h}}. \qquad (218)$$

in which α is a constant depending on the mode of vibration. For the three lowest modes the values of this constant are

$$\alpha_1 = 14.10, \qquad \alpha_2 = 20.56, \qquad \alpha_3 = 23.91.$$

The corresponding modes of vibration are represented by their nodal lines in Fig. 205 below.

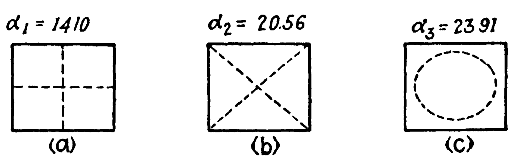

Fig. 205.

An extensive study of the nodal lines for this case and a comparison with experimental data are given in the paper by W. Ritz mentioned above.

* Loc. cit., p. 424.

From eq. (218) some general conclusions can be drawn which hold also in other cases of vibration of plates, namely,

(*a*) The period of the vibration of any natural mode varies with the square of the linear dimensions, provided the thickness remains the same;

(*b*) If all the dimensions of a plate, including the thickness, be increased in the same proportion, the period increases with the linear dimensions;

(*c*) The period varies inversely with the square root of the modulus of elasticity and directly as the square root of the density of material.

Vibration of a Circular Plate.—The problem of the vibration of a circular plate has been solved by G. Kirchhoff * who calculated also the frequencies of several modes of vibration for a plate with free boundary. The exact solution of this problem involves the use of Bessel functions. In the following an approximate solution is developed by means of the Rayleigh-Ritz method, which usually gives for the lowest mode an accuracy sufficient for practical applications. In applying this method it will be useful to transform the expressions (210) and (211) for the potential and kinetic energy to polar coordinates. By taking the coordinates as shown in Fig. 206, we see from the elemental triangle *mns* that by giving to the coordinate x a small increase dx we obtain

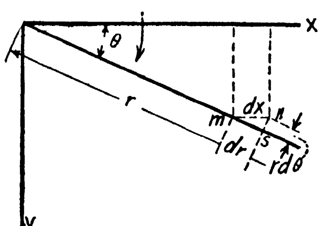

Fig. 206.

$$dr = dx \cos \theta; \qquad d\theta = -\frac{dx \sin \theta}{r}.$$

Then, considering the deflection v as a function of r and θ we obtain,

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial v}{\partial r}\cos \theta - \frac{\partial v}{\partial \theta}\frac{\sin \theta}{r}.$$

In the same manner we will find

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\sin \theta + \frac{\partial v}{\partial \theta}\frac{\cos \theta}{r}.$$

* See Journal f. Math. (Crelle), Vol. 40 (1850), or Gesammelte Abhandlungen von G. Kirchhoff, Leipzig 1882, p. 237, or Vorlesungen über math. Physik, Mechanik Vorlesung 30.

Repeating the differentiation we obtain

$$\frac{\partial^2 v}{\partial x^2} = \left(\frac{\partial}{\partial r}\cos\theta - \frac{\partial}{\partial\theta}\frac{\sin\theta}{r}\right)\left(\frac{\partial v}{\partial r}\cos\theta - \frac{\partial v}{\partial\theta}\frac{\sin\theta}{r}\right)$$

$$= \frac{\partial^2 v}{\partial r^2}\cos^2\theta - 2\frac{\partial^2 v}{\partial\theta\partial r}\frac{\sin\theta\cos\theta}{r} + \frac{\partial v}{\partial r}\frac{\sin^2\theta}{r} \tag{g}$$

$$+ 2\frac{\partial v}{\partial\theta}\frac{\sin\theta\cos\theta}{r^2} + \frac{\partial^2 v}{\partial\theta^2}\frac{\sin^2\theta}{r^2},$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 v}{\partial r^2}\sin^2\theta + 2\frac{\partial^2 v}{\partial\theta\partial r}\frac{\sin\theta\cos\theta}{r} + \frac{\partial v}{\partial r}\frac{\cos^2\theta}{r}$$

$$- 2\frac{\partial v}{\partial\theta}\frac{\sin\theta\cos\theta}{r^2} + \frac{\partial^2 v}{\partial\theta^2}\frac{\cos^2\theta}{r^2}, \tag{h}$$

$$\frac{\partial^2 v}{\partial x\partial y} = \frac{\partial^2 v}{\partial r^2}\sin\theta\cos\theta + \frac{\partial^2 v}{\partial r\partial\theta}\frac{\cos 2\theta}{r} - \frac{\partial v}{\partial\theta}\frac{\cos 2\theta}{r^2}$$

$$- \frac{\partial v}{\partial r}\frac{\sin\theta\cos\theta}{r} - \frac{\partial^2 v}{\partial\theta^2}\frac{\sin\theta\cos\theta}{r^2}, \tag{k}$$

from which we find

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2},$$

$$\frac{\partial^2 v}{\partial x^2}\frac{\partial^2 v}{\partial y^2} - \left(\frac{\partial^2 v}{\partial x\partial y}\right)^2 = \frac{\partial^2 v}{\partial r^2}\left(\frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2}\right) - \left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial v}{\partial\theta}\right)\right\}^2.$$

Substituting in eq. (210) and taking the origin at the center of the plate we obtain

$$V = \frac{D}{2}\int_0^{2\pi}\int_0^a\left\{\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2}\right)^2 - 2(1-\nu)\frac{\partial^2 v}{\partial r^2}\left(\frac{1}{r}\frac{\partial v}{\partial r}\right.\right.$$

$$\left.\left. + \frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2}\right) + 2(1-\nu)\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial v}{\partial\theta}\right)\right\}^2\right\} r\,d\theta\,dr, \tag{219}$$

where a denotes the radius of the plate.

When the deflection of the plate is symmetrical about the center, v will be a function of r only and eq. (219) becomes

$$V = \pi D\int_0^a\left\{\left(\frac{d^2 v}{dr^2} + \frac{1}{r}\frac{dv}{dr}\right)^2 - 2(1-\nu)\frac{d^2 v}{dr^2}\cdot\frac{1}{r}\frac{dv}{dr}\right\} r\,dr. \tag{220}$$

In the case of a plate clamped at the edge, the integral

$$\int\int\left[\frac{\partial^2 v}{\partial r^2}\left(\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}\right)-\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial v}{\partial \theta}\right)\right\}^2\right]rdrd\theta$$

vanishes and we obtain from (219)

$$V=\frac{D}{2}\int_0^{2\pi}\int_0^a\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}\right)^2 rd\theta dr. \tag{221}$$

If the deflection of such a plate is symmetrical about the center, we have

$$V=\pi D\int_0^a\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}\right)^2 rdr. \tag{222}$$

The expression for the kinetic energy in polar coordinates will be

$$T=\frac{\gamma h}{2g}\int_0^{2\pi}\int_0^a \dot{v}^2 rd\theta dr \tag{223}$$

and in symmetrical cases,

$$T=\frac{\pi\gamma h}{g}\int_0^a \dot{v}^2 rdr. \tag{224}$$

By using these expressions for the potential and kinetic energy the frequencies of the natural modes of vibration of a circular plate for various particular cases can be calculated.

Circular Plate Clamped at the Boundary.—The problem of the circular plate clamped at the edges is of practical interest in connection with the application in telephone receivers and other devices. In using the Rayleigh-Ritz method we assume

$$v = v_0 \cos pt, \tag{l}$$

where v_0 is a function of r and θ

In the case of the lowest mode of vibration the shape of the vibrating plate is symmetrical about the center of the plate and v_0 will be a function of r only. By taking v_0 in the form of a series like

$$v_0 = a_1\left(1-\frac{r^2}{a^2}\right)^2 + a_2\left(1-\frac{r^2}{a^2}\right)^3 + \cdots, \tag{m}$$

the condition of symmetry will be satisfied. The conditions at the boundary also will be satisfied because each term of the series (m) together with its first derivative vanishes when $r = a$.

The differential equation (217) in the case under consideration becomes

$$\frac{\partial}{\partial a_n}\int_0^a \left\{\left(\frac{d^2v_0}{dr^2} + \frac{1}{r}\frac{dv_0}{dr}\right)^2 - \frac{p^2\gamma h}{gD}v_0^2\right\}rdr = 0. \tag{225}$$

By taking only one term of the series (m) and substituting it in (225) we obtain,

$$\frac{96}{9a^2} - \frac{p^2\gamma h}{gD}\frac{a^2}{10} = 0,$$

from which

$$p = \frac{10.33}{a^2}\sqrt{\frac{gD}{\gamma h}}. \tag{226}$$

In order to get a closer approximation we take the two first terms of the series (m), then

$$\int_0^a \left(\frac{d^2v_0}{dr^2} + \frac{1}{r}\frac{dv_0}{dr}\right)^2 rdr = \frac{96}{9a^2}\left(a_1^2 + \frac{3}{2}a_1a_2 + \frac{9}{10}a_2^2\right),$$

$$\int_0^a v_0^2 rdr = \frac{a^2}{10}\left(a_1^2 + \frac{5}{3}a_1a_2 + \frac{5}{7}a_2^2\right).$$

Equations (225) become

$$\begin{aligned} a_1\left(\frac{192}{9} - \frac{x}{5}\right) + a_2\left(\frac{144}{9} - \frac{x}{6}\right) &= 0, \\ a_1\left(\frac{144}{9} - \frac{x}{6}\right) + a_2\left(\frac{96}{5} - \frac{x}{7}\right) &= 0, \end{aligned} \tag{n}$$

where

$$x = a^4p^2\frac{\gamma h}{gD}. \tag{o}$$

Equating to zero the determinant of eqs. (n) we obtain

$$x^2 - \frac{204 \times 48}{5}x + 768 \times 36 \times 7 = 0,$$

from which

$$x_1 = 104.3; \qquad x_2 = 1854.$$

Substituting in (o) we obtain

$$p_1 = \frac{10.21}{a^2}\sqrt{\frac{gD}{\gamma h}}; \qquad p_2 = \frac{43.04}{a^2}\sqrt{\frac{gD}{\gamma h}}. \tag{227}$$

p_1 determines the second approximation to the frequency of the lowest mode of vibration of the plate and p_2 gives a rough approximation to the frequency of the second mode of vibration, in which the vibrating plate has one nodal circle. By using the same method the modes of vibration having nodal diameters can also be investigated.

In all cases the frequency of vibration will be determined by the equation

$$p = \frac{\alpha}{a^2}\sqrt{\frac{gD}{\gamma h}}, \tag{228}$$

the constant α of which for a given number s of nodal circles and of a given number n of nodal diameters is given in the table below.

s	$n = 0$	$n = 1$	$n = 2$
0	10 21	21 22	34 84
1	39 78	.....	.....
2	88 90	.....	.....

In the case of thin plates the mass of the air or of the liquid in which the plate vibrates may affect the frequency considerably. In order to take this into account in the case of the lowest mode of vibration, equation (228) above should be replaced by the following equation,*

$$p_1 = \frac{10.21}{a^2\sqrt{1+\beta}}\sqrt{\frac{gD}{\gamma h}}, \tag{229}$$

in which

$$\beta = .6689\,\frac{\gamma_1}{\gamma}\frac{a}{h}$$

and (γ_1/γ) is the ratio of the density of the fluid to the density of the material of the plate.

Taking, for instance, a steel plate of 7 inches diameter and 1/8 inch thick vibrating in water, we obtain

$$\beta = .6689 \times \frac{1}{7.8} \times 28 = 2.40; \qquad \frac{1}{\sqrt{1+\beta}} = .542.$$

* This problem has been discussed by H. Lamb, Proc. Roy. Soc. London, Vol. 98 (1921), p. 205.

The frequency of the lowest mode of vibration will be lowered to .542 of its original value.

Other Kinds of Boundary Conditions.—In all cases the frequencies of a vibrating circular plate can be calculated from eq. (228). The numerical values of the factor α are given in the tables below.

For a free circular plate with n nodal diameters and s nodal circles α has the following values:*

s	$n = 0$	$n = 1$	$n = 2$	$n = 3$
0	. . . . . .	. . . . .	5 251	12 23
1	9 076	20 52	35 24	52 91
2	38 52	59 86		

For a circular plate with its center fixed and having s nodal circles α has the following values†

$s =$	0	1	2	3
$\alpha =$	3.75	20 91	60 68	119.7

The frequencies of vibration having nodal diameters will be the same as in the case of a free plate.

The Effect of Stretching of the Middle Surface of the Plate.—In the previous theory it was assumed that the deflection of the plate is small in comparison with its thickness. If a vibrating plate is under considerable static pressure such that the deflection produced by this pressure is not small in comparison with the thickness of the plate, the stretching of the middle surface of the plate should be taken into account in calculating the frequency of vibration. Due to the resistance of the plate to such a stretching the rigidity of the plate and the frequency of vibration increase with the pressure acting on the plate.‡ In order to show how the stretching of the middle surface may affect the frequency, let us consider again the case of a circular plate clamped at the boundary and

* Poisson's ratio is taken equal to $\frac{1}{3}$.

† See paper by R. V. Southwell, Proc. Roy. Soc., A, Vol. 101 (1922), p. 133; $\nu = .3$ is taken in these calculations.

‡ Such an increase in frequency was established experimentally. See paper by J. H. Powell and J. H. T. Roberts, Proc. Phys. Soc. London, Vol. 35 (1923), p. 170.

assume that the deflection of the plate under a uniformly distributed pressure is given by the equation *

$$v_0 = a_1\left(1 - \frac{r^2}{a^2}\right)^2. \qquad (m)'$$

In addition to the displacements v_0 at right angles to the plate the points in the middle plane of the plate will perform radial displacements u which vanish at the center and at the clamped boundary of the plate. The unit elongation of the middle surface in a radial direction, due to the displacements v_0 and u, is

$$e_r = \frac{1}{2}\left(\frac{dv_0}{dr}\right)^2 + \frac{du}{dr}. \qquad (p)$$

The elongation in a circumferential direction will be,

$$e_t = \frac{u}{r}. \qquad (r)$$

For an approximate solution of the problem we assume that the radial displacements are represented by the following series:

$$u = r(a - r)\,(c_1 + c_2 r + c_3 r^2 + \cdots), \qquad (s)$$

each term of which satisfies the boundary conditions.

Taking only the first two terms in the series (s) and substituting (s) and (m), in eqs. (p) and (r) the strain in the middle surface will be obtained and the energy corresponding to the stretching of the middle surface can now be calculated as follows:

$$V_1 = \frac{\pi E h}{1 - \nu^2}\int_0^a (e_r^2 + e_t^2 + 2\nu e_r e_t) r dr = \frac{\pi E h a^2}{1 - \nu^2}\Big(.250 c_1^2 a^2$$

$$+ .1167 c_2^2 a^4 + .300 c_1 c_2 a^3 - .00846 c_1 a \frac{8a_1^2}{a^2} \qquad (t)$$

$$+ .00682 c_2 a^2 \frac{8a_1^2}{a^2} + .00477 \frac{64 a_1^4}{a^4}\Big).$$

* This equation represents the deflections when the stretching of the middle surface is neglected. It can be used also for approximate calculation of the effect of the stretching.

Determining the constants c_1 and c_2 so as to make V_1 a minimum, we get, from the equations

$$\frac{\partial V_1}{\partial c_1} = 0, \qquad \frac{\partial V_1}{\partial c_2} = 0,$$

$$c_1 = 1.185\frac{a_1^2}{a^3}; \qquad c_2 = -1.75\frac{a_1^2}{a^4}.$$

Substituting in eq. (t):

$$V_1 = 2.59\pi D\frac{a_1^4}{a^2h^2}.$$

Adding this energy of stretching to the energy of bending (eq. 222) we obtain,

$$V = \frac{32}{3}\pi D\frac{a_1^2}{a^2} + 2.59\pi D\frac{a_1^4}{a^2h^2} = \frac{32}{3}\pi D\frac{a_1^2}{a^2}\left(1 + .244\frac{a_1^2}{h^2}\right). \qquad (u)$$

The second term in the brackets represents the correction due to the extension of the middle surface of the plate. It is easy to see that this correction is small and can be neglected only when the deflection a_1 at the center of the plate is small in comparison with the thickness h. The static deflection of the plate under the action of a uniformly distributed pressure w can now be found from the equation of virtual displacements,

$$\frac{\partial V}{\partial a_1}\delta a_1 = 2\pi w\delta a_1\int_0^a\left(1 - \frac{r^2}{a^2}\right)^2 rdr = \delta a_1\frac{\pi wa^2}{3}$$

from which

$$a_1 = \frac{wa^4}{64D}\cdot\frac{1}{1 + .488\dfrac{a_1^2}{h^2}}. \qquad (230)$$

The last factor on the right side represents the effect of the stretching of the middle surface. Due to this effect the deflection a_1 is no longer proportional to w and the rigidity of the plate increases with the deflection. Taking, for instance, $a_1 = \frac{1}{2}h$, we obtain, from (230)

$$a_1 = .89\frac{wa^4}{64D}.$$

The deflection is 11% less than that obtained by neglecting the stretching of the middle surface.

From the expression (u) of the potential energy, which contains not only the square but also the fourth power of the deflection a_1, it can be concluded at once that the vibration of the plate about its flat configuration will not be *isochronic* and the frequency will increase with the amplitude of vibration. Consider now small vibrations of the plate about a bent position given by eq. $(m)'$. This bending is supposed to be due to some constant uniformly distributed static pressure w. If Δ denotes the amplitude of this vibration, the increase in the potential energy of deformation due to additional deflection of the plate will be obtained from eq. (u) and is equal to *

$$\delta V = \frac{32}{3}\frac{\pi D}{a^2}\left(2a_1\Delta + \Delta^2 + \frac{.244}{h^2}(4a_1{}^3\Delta + 6a_1{}^2\Delta^2)\right).$$

The work done by the constant pressure w during this increase in deflection is

$$\delta W = \frac{\pi a^2 w\Delta}{3} = \frac{\pi a^2\Delta}{3}\cdot\frac{64a_1D\left(1 + .488\dfrac{a_1{}^2}{h^2}\right)}{a^4}.$$

The complete change in the potential energy of the system will be

$$\delta V - \delta W = \frac{32}{3}\frac{\pi D\Delta^2}{a^2}\left(1 + \frac{1.464a_1{}^2}{h^2}\right).$$

Equating this to the maximum kinetic energy,

$$T_{\max} = \frac{\pi\Delta^2 p^2\gamma h}{g}\int_0^a\left(1 - \frac{r^2}{a^2}\right)^4 r dr = \frac{\pi\Delta^2 a^2\gamma h}{10g}p^2$$

we obtain

$$p = \frac{10.33}{a^2}\sqrt{\frac{gD}{\gamma h}}\sqrt{1 + 1.464\frac{a_1{}^2}{h^2}}. \tag{231}$$

Comparing this result with eq. (226) it can be concluded that the last factor on the right side of eq. (231) represents the correction due to the stretching of the middle surface of the plate.

It should be noted that in the above theory equation $(m)'$ for the deflection of the plate was used and the effect of tension in the middle surface of the plate on the form of the deflection surface was neglected. This is the reason why eq. (231) will be accurate enough only if the deflections are not large, say $a_1 \leq h$. Otherwise the effect of tension in the middle surface on the form of the deflection surface must be taken into consideration.

* Terms with Δ^3 and Δ^4 are neglected in this expression.

71. Vibration of Turbine Discs.—*General.*—It is now fairly well established that fractures which occur in turbine discs and which cannot be explained by defects in the material or by excessive stresses due to centrifugal forces may be attributed to flexural vibrations of these discs. In this respect it may be noted that direct experiments have shown * that such vibrations, at certain speeds of the turbine, become very pronounced and produce considerable additional bending stresses which may result in fatigue of the metal and in the gradual development of cracks, which usually start at the boundaries of the steam balance holes and other discontinuities in the web of the turbine disc, where stress concentration is present.

There are various causes which may produce these flexural vibrations in turbine discs but the most important is that due to non-uniform steam pressure. A localized pressure acting on the rim of a rotating disc is sufficient at certain speeds to maintain lateral vibrations in the disc and experiments show that the application of a localized force of only a few pounds, such as produced by a small direct current magnet to the side of a rotating turbine disc makes it respond violently at a whole series of critical speeds.

Assume now that there exists a certain irregularity in the nozzles which results in a non-uniform steam pressure and imagine that a turbine disc is rotating with a constant angular velocity ω in the field of such a pressure. Then for a certain spot on the rim of the disc the pressure will vary with the angle of the rotation of the wheel and this may be represented by a periodic function, the period of which is equal to the time of one revolution of the disc. In the most general case such a function may be represented by a trigonometrical series

$$w = a_0 + a_1 \sin \omega t + a_2 \sin 2\omega t + \cdots b_1 \cos \omega t + b_2 \cos 2\omega t + \cdots.$$

By taking only one term of the series such as $a_1 \sin \omega t$ we obtain a periodic disturbing force which may produce large lateral vibration of the disc if the frequency $\omega/2\pi$ of the force coincides with one of the natural frequencies $p/2\pi$ of the disc. From this it can be appreciated that the calculation of the natural frequencies of a disc may have a great practical importance.

A rotating disc, like a circular plate, may have various modes of vibration which can be sub-divided into two classes:

* See paper by Wilfred Campbell, Trans. Am. Soc. Mech. Eng., Vol. 46 (1924), p. 31. See also paper by Dr. J. von Freudenreich, Engineering, Vol. 119, p. 2 (1925).

a. Vibrations symmetrical with respect to the center, having nodal lines in the form of concentric circles, and

b. Unsymmetrical having diameters for nodal lines. The experiments show that the symmetrical type of vibration very seldom occurs and no disc failure can be attributed to this kind of vibration.

In discussing the unsymmetrical vibrations it can be assumed that the deflection of the disc has the following form,

$$v = v_0 \sin n\theta \cos pt, \tag{a}$$

in which, as before, v_0 is a function of the radial distance r only, θ determines the angular position of the point under consideration, and n represents the number of nodal diameters.

The deflection can be taken also in the form

$$v = v_0 \cos n\theta \sin pt. \tag{a)'$$

Combining (a) and $(a)'$ we obtain

$$v = v_0(\sin n\theta \cos pt \pm \cos n\theta \sin pt) = v_0 \sin (n\theta \pm pt),$$

which represents traveling waves. The angular speed of these waves traveling around the disc will be found from the condition

$$n\theta \pm pt = \text{const.}$$

From

$$\theta = \pm \frac{p}{n} t + \text{const.}$$

we obtain two speeds $- p/n$ and $+ p/n$ which are the speeds of the backward and forward traveling waves, respectively. The experiments of Campbell * proved the existence of these two trains of waves in a rotating disc and showed also that the amplitudes of the backward moving waves are usually larger than those of the forward moving waves. Backward moving waves become especially pronounced under conditions of resonance when the backward speed of these waves in the disc coincides exactly with the forward angular velocity of the rotating disc so that the waves become *stationary in space.* The experiments show that this type of vibration is responsible in a majority of cases for disc failures.

Calculation of the Frequencies of Disc Vibrations.—In calculating the frequencies of the various modes of vibration of turbine discs the Rayleigh-

* Loc. cit., p. 435.

Ritz method is very useful.* In applying this method we assume that the deflection of the disc has a form

$$v = v_0 \sin n\theta \cos pt. \qquad (a)''$$

In the particular case of vibration symmetrical with respect to the center the deflection will be:

$$v = v_0 \cos pt. \qquad (b)$$

Considering in the following this particular case the maximum potential energy of deformation will be, from eq. (220),

$$V_{\max} = \pi \int_b^a D \left\{ \left(\frac{d^2 v_0}{dr^2} + \frac{1}{r}\frac{dv}{dr} \right)^2 - 2(1 - \nu) \frac{d^2 v}{dr^2} \frac{1}{r} \frac{dv}{dr} \right\} rdr, \qquad (c)$$

where a, b are outer and inner radii of the disc,

$D = \dfrac{Eh^3}{12(1 - \nu^2)}$ is flexural ridigity of the disc, which in this case will be variable due to variation in thickness h of the disc.

In considering the vibration of a rotating disc not only the energy of deformation but also the energy corresponding to the work done during deflection by the centrifugal forces must be taken into consideration. It is easy to see that the centrifugal forces resist any deflection of the disc and this results in an increase in the frequency of its natural vibration. In calculating the work done by the centrifugal forces let us take an element cut out from the disc by two cylindrical surfaces of the radii r and $r + dr$ (Fig. 207). The radial displacement of this element towards the center due to the deflection will be

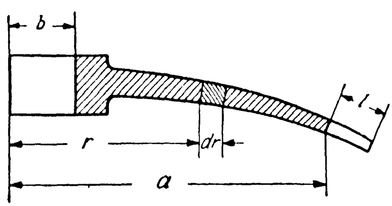

FIG. 207.

$$\frac{1}{2} \int_b^r \left(\frac{dv_0}{dr} \right)^2 dr.$$

The mass of the element is

$$\frac{2\pi r h \gamma}{g} dr$$

* The vibration of turbine discs by using this method was investigated by A. Stodola, Schweiz. Bauz., Vol. 63, p. 112 (1914).

and the work done during the deflection by the centrifugal forces acting on this element will be

$$-\frac{2\pi r^2\omega^2 h\gamma}{g}dr\cdot\frac{1}{2}\int_b^r\left(\frac{dv_0}{dr}\right)^2 dr. \qquad (d)$$

The energy corresponding to the work of the centrifugal forces will be obtained by summation of such elements as (d) in the following form,

$$(V_1)_{\max} = \int_b^a \frac{\pi r^2\omega^2 h\gamma}{g}dr\int_b^r\left(\frac{dv_0}{dr}\right)^2 dr. \qquad (e)$$

The maximum kinetic energy is given by the equation

$$T = \int_b^a \frac{2\pi r\gamma h}{2g}\dot{v}^2 dr.$$

Substituting expression (b) for v we obtain

$$T_{\max} = \frac{\pi\gamma p^2}{g}\int_b^a hv_0^2 r dr. \qquad (f)$$

Now, from the equation

$$V_{\max} + (V_1)_{\max} = T_{\max}$$

we deduce

$$p^2 = \frac{V_{\max} + (V_1)_{\max}}{\dfrac{\pi\gamma}{g}\displaystyle\int_b^a hv_0^2 r dr}. \qquad (g)$$

In order to obtain the frequency the deflection curve v_0 should be chosen so as to make the expression (g) a minimum. This can be done graphically by assuming for v_0 a suitable curve from which v_0, dv_0/dr and d^2v_0/dr^2 can be taken for a series of equidistant points and then the expressions (c), (e) and (f) can be calculated. By gradual changes in the shape of the curve for v_0 a satisfactory approximation for the lowest frequency can be obtained * from eq. (g).

In order to take into account the effect of the blades on the frequency of natural vibration the integration in the expression (e) and (f) for the

* Such a graphical method has been developed by A. Stodola, loc. cit., p. 437. It was applied also by E. Oehler, V. D. I., Vol. 69 (1925), p. 335, and gave good agreement with experimental data.

work done by the centrifugal forces and for the kinetic energy must be extended from b to $a + l$ where l denotes the length of the blade. In this calculation the blades can be assumed to be straight during vibration of the disc so that no addition to the expression for the potential energy (c) will be necessary.

In an analytical calculation of the lowest frequency of a vibrating disc we take v_0 in the form of a series such as

$$v_0 = a_1(r - b)^2 + a_2(r - b)^3 + a_3(r - b)^4 + \cdots,$$

which satisfies the conditions at the built-in inner boundary of the disc, where v_0 and dv_0/dr become equal to zero. The coefficients a_1, a_2, a_3 should now be chosen so as to make expression (g) a minimum. Proceeding as explained in the previous article (see p. 429) a system of equations analogous to the equations (225) and linear in a_1, a_2, a_3 $\cdots$ can be obtained. Equating to zero the determinant of these equations, the frequency equation will be found.

In the case of a mode of vibration having diameters as nodal lines the expression (a)'' instead of (b) must be used for the deflections. The potential energy will be found from eq. (219): it is only necessary to take into consideration that in the case of turbine discs the thickness and the flexural rigidity D are varying with the radial distance r so that D must be retained under the sign of integration. Without any difficulty also the expressions for V_1 and T can be established for this case and finally the frequency can be calculated from eq. (g) exactly in the same manner as it was explained above for the case of a symmetrical mode of vibration.*

When the disc is stationary V_1 vanishes and we obtain from equation (g)

$$p_1^2 = \frac{V_{\max}}{\dfrac{\pi\gamma}{g}\displaystyle\int_b^a hv_0^2 r\,dr}, \qquad (g)'$$

which determines the frequency of vibration due to elastic forces alone.

Another extreme case is obtained when the disc is very flexible and the restoring forces during vibration are due entirely to centrifugal forces. Such conditions are encountered, for instance, when experimenting with

* The formulae for this calculation are developed in detail by A. Stodola, loc. cit.

flexible discs made of rubber. The frequency will be determined in this case from eq.

$$p_2^2 = \frac{(V_1)_{\max}}{\frac{\pi\gamma}{g}\int_b^a hv_0^2 r\,dr}. \qquad (g)''$$

Now, from eq. (g), we have

$$p^2 = p_1^2 + p_2^2. \qquad (h)$$

If the frequencies p_1 and p_2 are determined in some way, the resulting frequency of vibration of the disc will be found from eq. (h). In the case of discs of constant thickness and fixed at the center an exact solution for p_1 and p_2 has been obtained by R. V. Southwell.* He gives for p_1^2 the following equation,

$$p_1^2 = \frac{\alpha}{a^4}\frac{Dg}{\gamma h}. \qquad (k)$$

The values of the constant α for a given number n of nodal diameters and a given number s of nodal circles are given in the table below.†

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
$s = 0$	14.1	0	29.0	156
$s = 1$	438	422	1210	2840

The equation for calculating p_2^2 is

$$p_2^2 = \lambda\omega^2, \qquad (l)$$

in which ω is the angular velocity and λ is a constant given in the table below,

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
$s = 0$	0	1	2.35	4.05
$s = 1$	3.3	5.95	8.95	12.3

* Loc. cit., p. 431.

† All other notations are the same as for circular plates (see p. 428). Poisson's ratio is taken equal to .3 in these calculations.

Determining p_1^2 and p_2^2 from the equations (k) and (l) the frequency of vibration of the rotating disc will then be found from eq. (h).*

In the above theory of the vibration of discs the effect of non-uniform heating of the disc was not considered. In a turbine in service the rim of the disc will be warmer than the web. Due to this factor compressive stresses in the rim and tensile stresses in the web will be set up which may affect the frequencies of the natural vibrations considerably. The experiments and calculations† show that for vibrations with 0 and 1 nodal diameters, the frequency is increased, whereas with a larger number of nodal diameters, the frequency is lowered by such a non-uniform heating.

* A discussion of the differential equation of vibration for the case of a disc of variable thickness is given in the paper by Dr. Fr. Dubois, Schweiz. Bauz., Vol. 89, p. 149 (1927).

† Freudenreich, loc. cit., p. 435.

APPENDIX

VIBRATION MEASURING INSTRUMENTS

1. General.—Until quite recently practical vibration problems in the shops and in the field were usually left to the care of men who did not have great knowledge of the theory of vibration and based their opinions on data obtained from experience and gathered by the unaided senses of touch, sight and hearing. With the increasing dimensions and velocities of modern rotating machinery, the problem of eliminating vibrations becomes more and more important and for a successful solution of this problem the compilation of *quantitative* data on the vibrations of such machines and their foundations becomes necessary. Such quantitative results, however, can be got only by means of *instruments*. The fundamental data to be measured in investigating this problem are: (*a*) the frequency of the vibration, (*b*) its amplitude, (*c*) the type of wave, simple harmonic, or complex, and (*d*) the stresses produced by this vibration.

Modern industry developed many instruments for measuring the above quantities and in the following some of the most important, which have found wide application, will be described.*

2. Frequency Measuring Instruments.—A knowledge of the frequency of a vibration is very important and often gives a valuable clue to its source. The description of a very simple frequency meter, *Frahm's tachometer*, which has long been used in turbo generators, was given before. (See page 27.) The *Fullarton vibrometer* is built on the same principle. It is shown in Fig. 208. This instrument consists of a claw A to be clamped under a bolt head, two joints B rotatable at right angles to each other, a main frame bearing a reed C, a length scale D on the side, an amplitude scale E across the top, and a long screw F. A clamp carriage rides on the main frame, its position being adjusted by the screw.

* See the paper by J. Ormondroyd, Journal A.I.E.E., Vol. 45 (1926), p. 330. See also the paper by P. A. Borden, A.I.E.E. Trans., 1925, p. 238, and the paper by H. Steuding, V.D.I., Vol. 71 (1927), p. 605, representing an abstract from a very complete investigation on vibration recording instruments made for the Special Committee on Vibration organized by the V.D.I. (Society of German Engineers).

The reed is held tightly in a fixed clamp at the bottom of the frame and its free length is varied by the position of the movable clamp on the carriage. The instrument is bolted to the vibrating machine* and the free length of the reed is adjusted until the largest amplitude of motion is obtained at the end of the reed. This is read on the transverse scale. The instrument then is in resonance with the impressed frequency. This frequency can be determined by measuring the free length of the reed.

This device is so highly selective (damping forces extremely small) that it can be used only on vibrations with almost absolutely constant

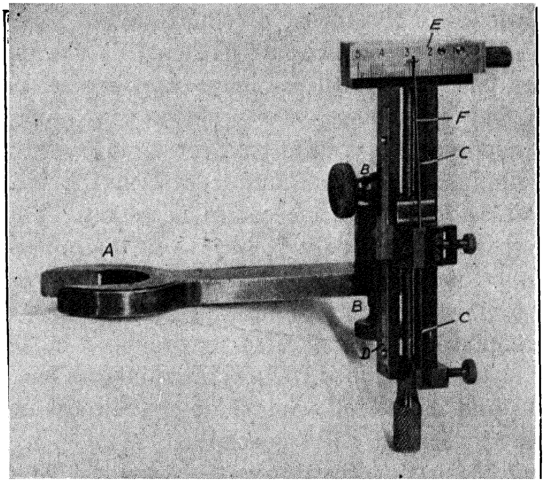

FIG. 208.

frequency. The least variation in frequency near the resonance point will give a very large fluctuation in amplitude. This limits the instrument to uses on turbo generators and other machinery in which the speed varies only slightly.

3. The Measurement of Amplitudes.—There are many instances where it is important to measure only the amplitude of the vibration. This is true in most cases of studying forced periodic vibrations of a known frequency such as are found in structures or apparatus under the action of rotating machinery. Probably the most frequent need for measuring amplitudes occurs in power plants, where vibrations of the

* The weight of the machine should be considerably larger than the weight of the instrument to exclude the possibility of the instrument affecting the motion of the machine.

building, of the floor, of the foundation or of the frame are produced by impulses given once, a revolution due to unbalance of the rotating parts.

The theory on which seismic instruments are based is given on page 19. An amplitude meter on this principle, built by the Vibration Specialty Company of Philadelphia, is shown in Fig. 209. The photograph shows the instrument with the side cover off. It is of the seismographic type. A steel block (1) is suspended on springs (3) in a heavy frame (2), the

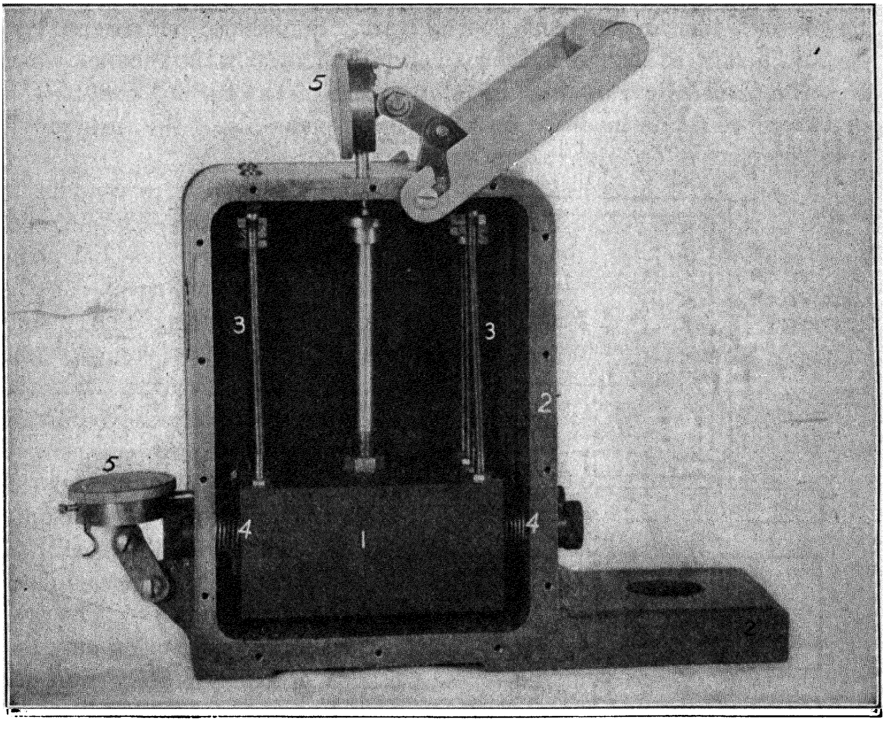

Fig. 209.

additional compression springs (4) centering the block horizontally. The frequencies of the natural vibrations of the block both in vertical and horizontal direction are about 200 per minute. The frame carries two dial indicators (5), the plungers of which touch the block. The instrument is to be bolted to the structure under investigation. The frequency of vibration produced by high speed rotating machinery is usually several times higher than the natural frequency of the vibrometer and the block of the instrument can be considered as stationary in space. The indicators register the vertical and horizontal components of the relative motion between the block and the frame, their hands moving back and forth over arcs giving the double amplitudes of these components.

This instrument proved to be very useful in power plants for studying the vibration of turbo generators. It is a well known fact that at times a unit, due probably to non-uniform temperature distribution in the rotor, begins to vibrate badly when brought to full speed, the vibrations persisting for a long period. This condition may be cured by slowing the machine down and then raising the speed again. Sometimes vibrations may be built up also at changes in the load or due to a drop in the vacuum, which is accompanied by variations in temperature of the turbine parts. One or two vibrometers mounted on the bearing pedestals of the turbine will give complete information about such vibrations.

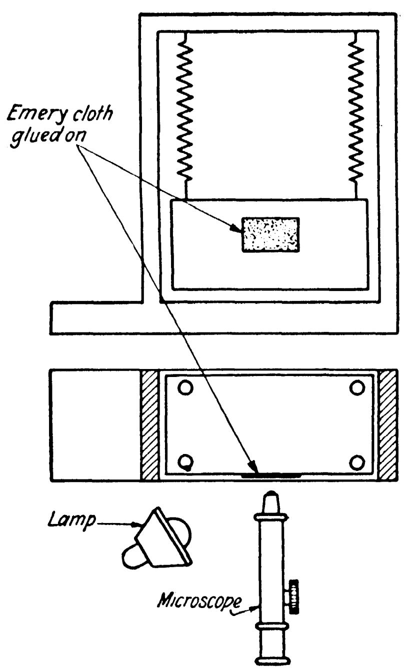

FIG. 210.

The instrument is also very useful for balancing the rotors at high speed, especially when a very fine balancing is needed. The elimination of the personal element during this operation is of great importance. The balancing takes a long time when the unit is in service, several days passing sometimes between two consecutive trials and a numerical record of the amplitude of vibration gives a definite method of comparing the condition of the machine for the various locations of balancing weights. The procedure of balancing by using only the amplitudes of the vibration was described before (see page 70).

Another interesting application of this instrument is shown in Fig. 210. With the front cover off the instrument, the actual path of a point on the vibrating pedestal of a turbo generator can be studied.* A piece of emery cloth of a medium grade is glued to the steel block of the instrument. A light is thrown onto the emery, giving very sharp point reflections on the

* This method was devised by Mr. G. B. Karelitz, Research Engineer of the Westinghouse Electric & Manufacturing Company.

crystals of carborundum. A microscope is rigidly attached to the pedestal under investigation and focused on the emery cloth. The block being stationary in space, the relative motion of the microscope and the cloth

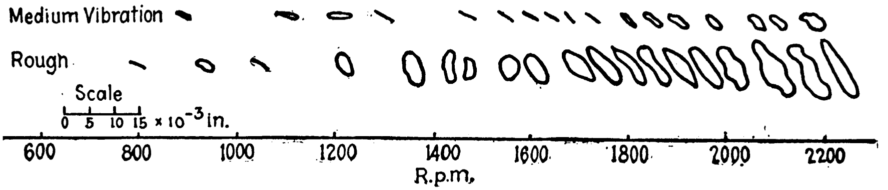

Fig. 211.

can be clearly seen, the points of light scribing bright figures, of the same kind as the well known Lissajous' figures. Typical figures as obtained on a pedestal of an 1800 r.p.m. turbine are shown in Fig. 211.

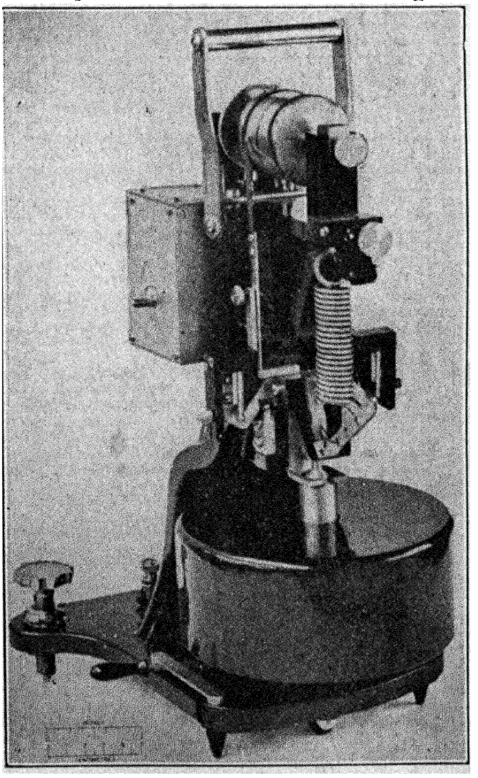

Fig. 212.

4. Seismic Vibrographs.—Seismic vibrographs are used where a complete analysis of the vibration is required. The chief application these instruments find is in the measurement of floor vibrations in buildings, vibrations of foundations of machines and vibrations of bridges. By analyzing a vibrograph record into simple harmonic vibrations, it is possible sometimes to find out the source of the disturbing forces producing these component vibrations.

The Vibrograph constructed by the Cambridge Instrument Company* is shown in Figs. 212 and 213. This instrument records vertical vibrations.

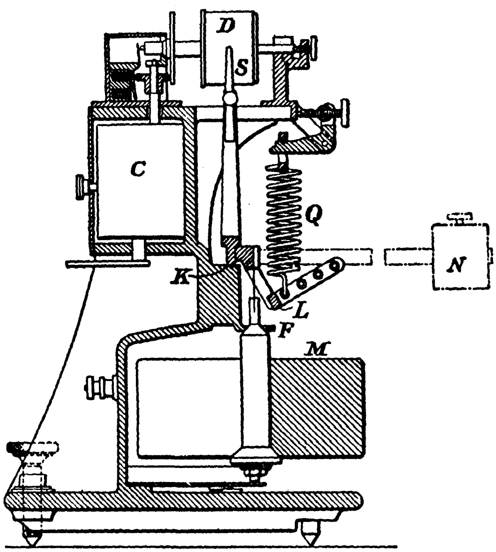

FIG. 213.

If required for violent oscillations, the instrument is fitted with a steel yard attachment indicated by the dotted lines of the sectional diagram, Fig. 213. The instrument consists of a weighted lever, pivoted on knife edges on a stand which, when placed on the structure or foundation, partakes of its vibrations. The small lever movements caused by the vibrations are recorded on a moving strip of celluloid by a fine point carried at the extremity of an arm joined to the lever. The heavy mass M is attached by a metal strip to a steel block which is pivoted to the stand by means of the knife edges K. The steel block forms a short lever, the

* For a more detailed description of this instrument, see Engineering, Vol. 119 (1925), p. 271.

effective length of which is equal to the horizontal distance between the strip and the knife edges. The weight M is balanced by a helical spring Q suspended from the upper portion of the stand. The lower end of this spring is hooked into one of the four holes in the arm of the bell crank lever L and by selecting one of these holes the natural frequency of the moving system can be altered. An arm extending upward from the pivoted steel block, previously referred to, has at its upper extremity a flat spring S, carrying the recording point. This point bears upon the surface of a celluloid film (actually a portion of clear moving picture film) wrapped around the split drum D which is rotated by means of the clockwork mechanism C. By means of an adjustable governor the speed of the film can be varied between about 4 mm. and 20 mm. per second. In the narrow gap between the two portions of the split drum D rests a second point which can be shifted laterally by means of an electromagnet acting through a small lever mechanism inside the drum. This electromagnet is connected to a separate clock, making contact every tenth of a second, or other time interval. Thus a zero line with time markings is recorded on the back of the film simultaneously with the actual "vibrogram" on the front. The records obtained can be read by a microscope accurately to .01 mm. and as the initial magnification of the recording instrument is 10, a vertical movement of the foundation of 10^{-4} cm. is clearly measurable.*

In Fig. 214, the "Geiger" Vibrograph is shown.† The whole instrument, the dimensions of which are about $8'' \times 6'' \times 6''$, has to be attached to the vibrating machine or structure. A heavy block on weak springs supported inside the instrument will remain still in space. The relative motion between this block and the frame of the instrument is transmitted to a capillary pen which traces a record of it on a band of paper, $2\frac{1}{2}''$ wide. A clockwork, which can be set at various speeds, moves the band of paper and rolls it up on a pulley. For time marking there is a cantilever spring attached to the frame with a steel knob and a pen on its end. This cantilever has a natural frequency of 25 cycles per second. It can be operated either by hand or electrically by means of two dry cells. It must be deflected every second or so and traces a damped 25-cycle wave on the record. The natural period of the seismographic mass itself is approximately $1\frac{1}{2}$ per second. The magnification of the lever

* This method of recording was first adopted by W. G. Collins in the Cambridge microindicator for high-speed engines, see Engineering, Vol. 13 (1922), p. 716. See also Trans. of the Optical Society, Vol. 27 (1925–1926), p. 215.

† For a more detailed description of this instrument, see V.D.I., Vol. 60 (1916), p. 811.

system connecting this mass with the pen is adjustable. Satisfactory records can be obtained with a magnification of 12 times for frequencies up to 130 per second. It will operate satisfactorily even to 200 cycles

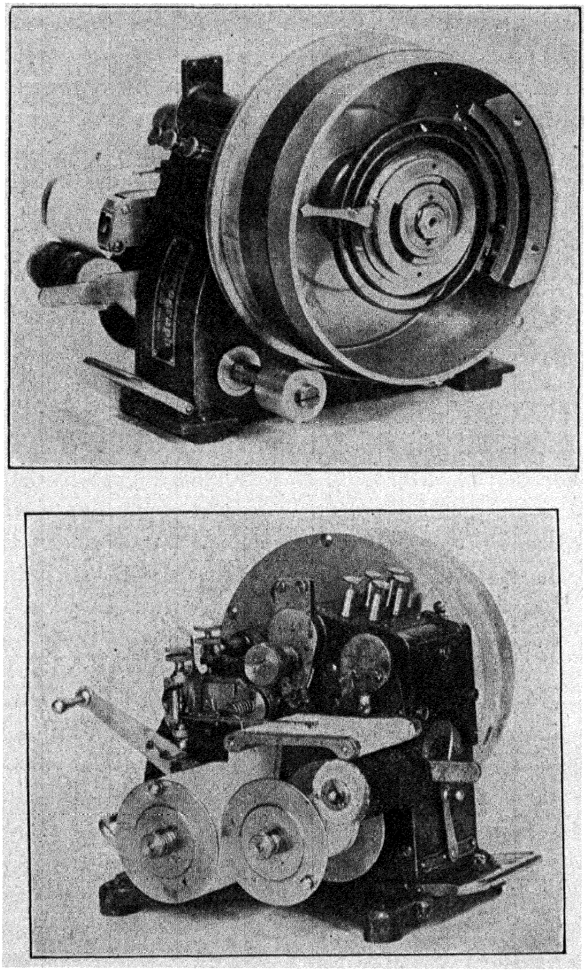

Fig. 214.

per second, provided the magnification chosen is not more than three times. It should be noted that by means of an adjustment at the seismographic mass it is possible to obtain a record of the vibration *in any direction.*

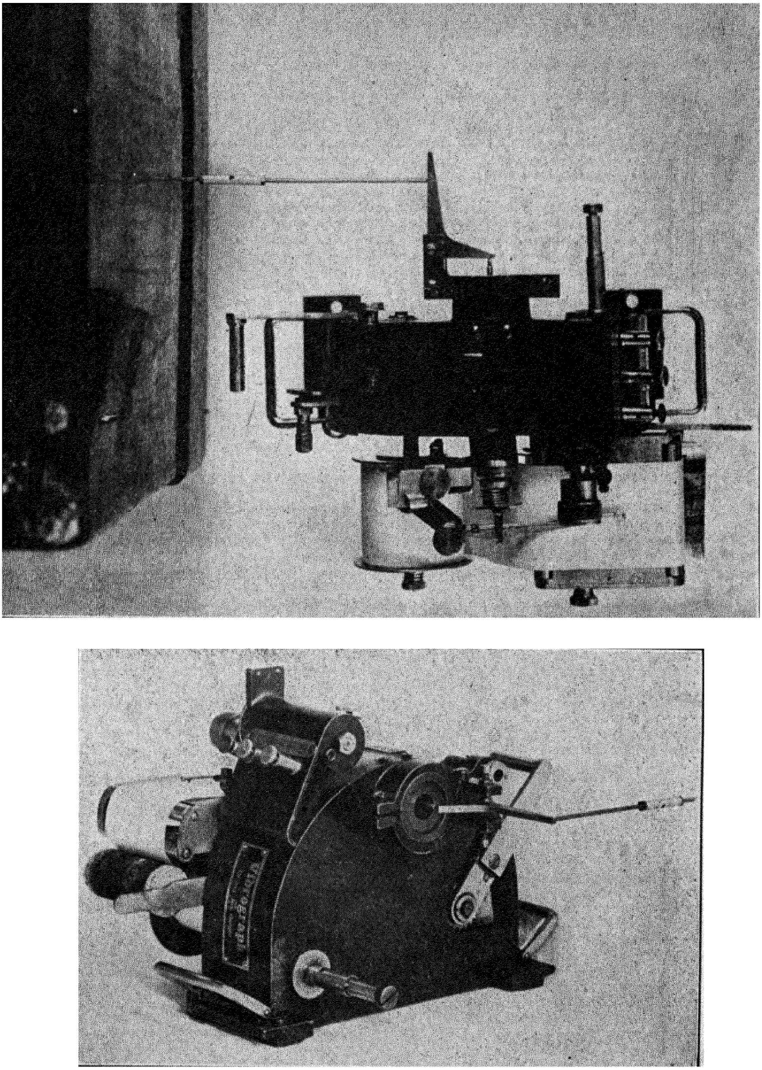

FIG. 215.

In cases where the vibrating body is so small that its vibration will be affected by the comparatively large mass of the instrument, it is possible to use it merely as a recorder ("universal recorder," as it is called by the inventor). The seismographic mass is then taken out of it and the instrument has to be supported immovable in space in some manner; for instance, by suspending it from a crane. The lever system of the recording pen is directly actuated by a tiny rod which touches the vibrating body. (Fig. 215.) With this arrangement, magnifications of 100 times at 60 cycles

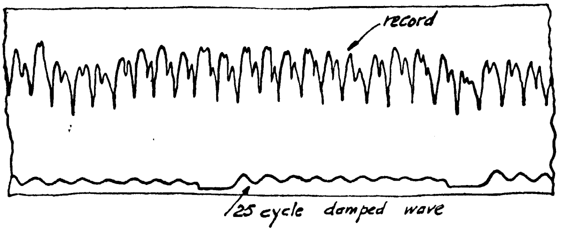

FIG. 216.

and of 15 times at 150 cycles can be obtained. A record of this instrument is shown in Fig. 216.

5. Torsiograph.—Many instruments have been designed for recording torsional vibrations in shafting. An instrument of this kind which has found a large application is shown in Fig. 217. This instrument, designed

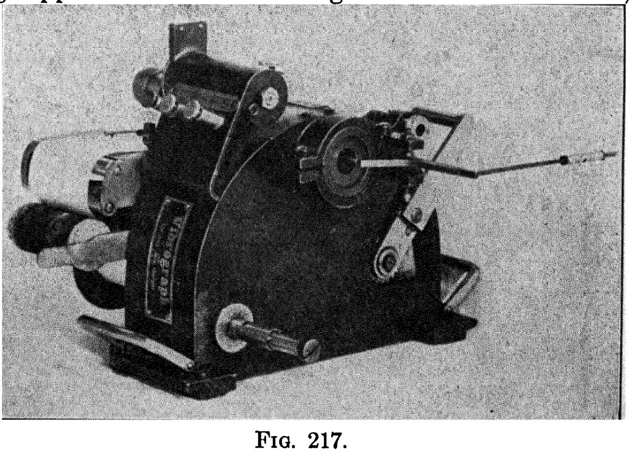

FIG. 217.

by A. Geiger, has the same recording and timing device as the vibrograph described above, but differs from it in its seismographic part. It has a light pulley of about 6″ diameter, in which a heavy fly-wheel is mounted concentrically and free to turn on the same axis. The connection between pulley and mass is by means of a very flexible spiral spring. The natural frequency of torsional oscillations of this mass, when the pulley is kept steady, is about $1\frac{1}{2}$ per second. In operation the pulley is driven by means of a short belt, 1″ wide, from the shaft of which the torsional oscillations are to be measured. The pulley moves with the shaft, but the heavy mass inside will revolve at practically uniform angular velocity provided that the frequency of torsional vibrations is above a certain value, say four times larger than the natural frequency of the instrument. The relative motion of the pulley and a point on the circumference of the fly-wheel is transmitted through a lever system to the recording pen. This instrument operates up to 200 cycles per second for low magnifications, and the magnification of the oscillatory motion on the circumference of the shaft can be made as high as 24 to 1 for low frequency motions. Small oscillations should be recorded from a portion of the shaft with as large a diameter as possible. Large oscillations should be measured on small diameter shafts to keep the record within the limits of the instrument. The limit to the size of the driving pulley is established by the effects of centrifugal forces on the spiral spring which is attached between the fly-wheel and the pulley. At about 1500 r.p.m. the centrifugal forces distort the spring enough to push the pen off the recording strip. This instrument has been successfully applied in studying torsional vibrations in Diesel engine installations such as in locomotives and submarines. Recently a combined torsiograph—vibrograph—universal recorder has been put on the market.

6. Torsion Meters.—There are cases where not only the oscillations of angular velocity as measured by Geiger's Torsiograph, but also the torque in a shaft transmitting power, is of interest. Many instruments have been designed for this purpose, especially in connection with measuring the power transmitted through propeller shafts of ships.* The generally accepted method is to measure the relative movement of two members fixed in two sections at a certain distance from each other on the shaft. The angle made by these members relative to each other is observed or recorded by an oscillograph. Knowing the speed of rotation of the shaft and its modulus of rigidity, the horse power transmitted can be determined. Fig. 218 represents the torsion meter designed by E. B. Moullin of the Engineering Laboratory, Cambridge, England.† "The

* There are various methods of measuring and recording the angle of twist in shafts, to be divided in four groups: (*a*) mechanical, (*b*) optical, (*c*) stroboscopic, and (*d*) electrical methods. Descriptions of the instruments built on these various principles are given in the paper by H. Steuding, mentioned above. (See page 443. See also the paper by V. Vieweg in the periodical "Der Betrieb," 1921, p. 378.

† See the paper by Robert S. Whipple, Journal of the Optical Soc. of America, Vol. 10 (1925), p. 455.

relative movement of the two members of the instrument is measured electrically and continuously throughout the revolution, so that the instrument can be fixed in the ship's tunnel on the shaft, and the observations made at a distance. The Moullin torsion meter has been used to measure the torque transmitted on ships' shafts up to 10 inches in diameter, and transmitting 1500 H.P. The instrument consists of an air-gap choker, one-half carried by a ring fixed to a point on the shaft, and the other half carried adjacent to the first but attached to a sleeve fixed to

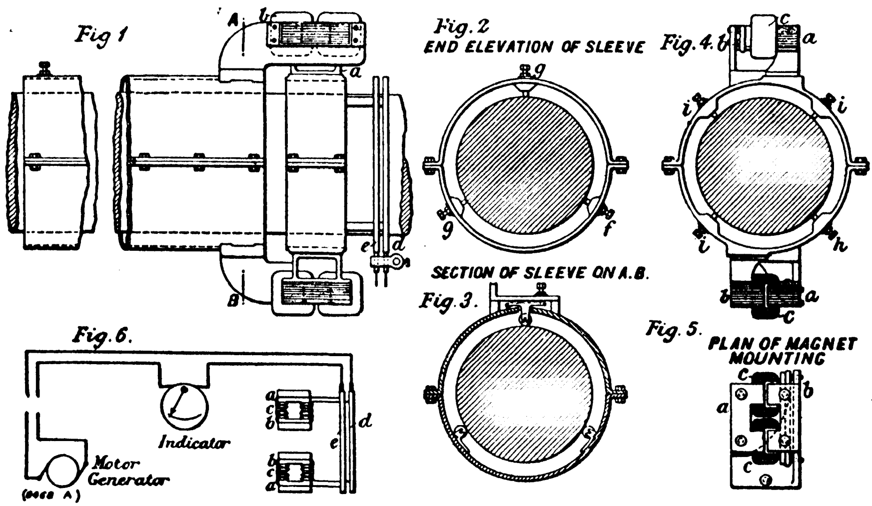

FIG. 218.

the shaft about four feet away. Fig. 218 shows the arrangement of the halves of the choker, of which the one *a* is fixed to the ring, and the other *b* is attached to the sleeve. A small alternating current generator supplies current to the windings *c* at a frequency of 60 cycles per second and about 100 volts. As the shaft twists, the gap opens for forward running (and closes in running astern) and the current increases in direct proportion to the gap so that the measurements on a record vary directly with the torque. Two chokers are fitted, one at each end of a diameter, so that they are in mechanical balance, and, being connected electrically in series, are unaffected by bending movements. Current is led in and out of the chokers by two slip rings *d* and *e*." By using a standard oscillograph a continuous record can be obtained such as shown in Fig. 219.

In Fig. 220 is shown the torsion meter of Amsler, which is largely used for measuring the efficiency of high speed engines.

The connecting flanges D and L of the torsion meter are usually keyed on to the ends 1 and 2 of the driving and the driven shafts. The elastic bar which transmits the torsional effort is marked G. It is fitted at the ends of the chucks F and H. The chuck F is always fastened to the

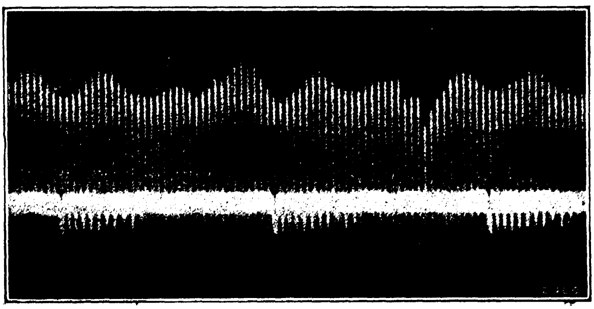

FIG. 219.

sleeve A on which the flange B is keyed. The flange B is bolted to the flange D, and the flange J to the flange L; the ends of the bar G are thus rigidly secured to the flanges D and L. In order to measure the angle of twist the discs M, N and O are used. M is fastened to the chuck J, while

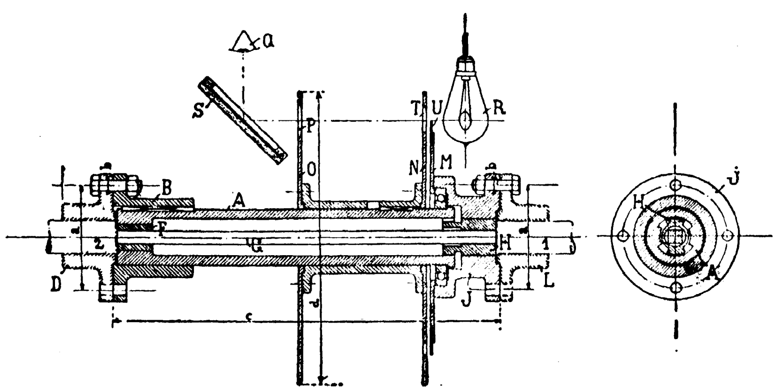

FIG. 220.

the other two, N and O, are fixed to the sleeve A. When the measuring bar G is twisted under the action of a torque, the disc M turns with respect to the other two discs N and O through a definite angle of twist. The edge U of the disc M is made of a ring of transparent celluloid on which a scale is engraved. Opposite this scale there is a small opening T in the disc N, and a fine radial slot which serves as a pointer for making readings on the scale. The disc O has no opening opposite T but only a radial slot like the one in the disc N, and through this the observer looks when reading the position of the indicator T on the scale U by means of the mirror S placed at an angle of 45 degrees to the visual ray. The scale engraved on the celluloid is well illuminated from behind by means of a lamp R. If the apparatus has a considerable velocity, say not less than 250 revolutions per minute, the number of luminous impressions per second will be sufficient to give the impression of a steady image and the reading of the angle of

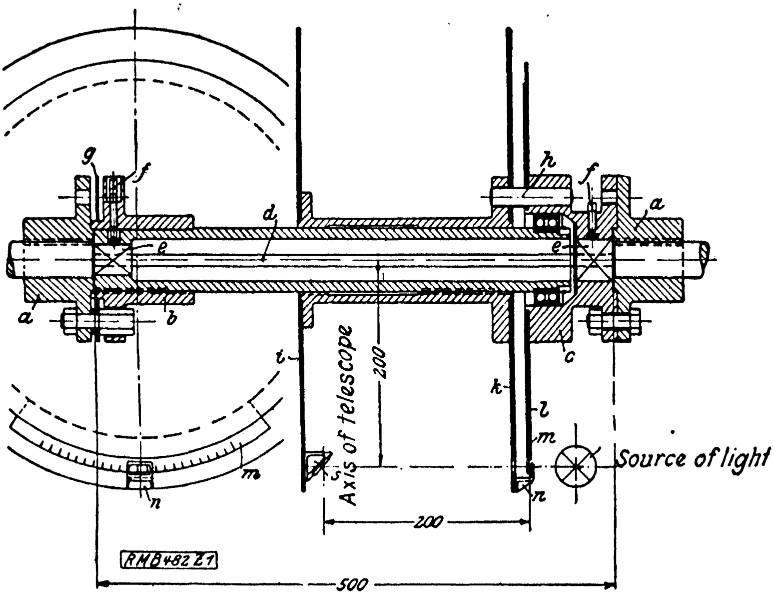

FIG. 221.

twist can be taken with a great accuracy, provided this angle remains constant during rotation. Knowing the angle of twist and the torsional rigidity of the bar G, the torque and the power transmitted can easily be calculated.

V. Vieweg improved the instrument described above by attaching the mirror S to the disc as shown in Fig. 221 and by taking the distance of this mirror from the scale mn equal to the distance of the mirror from the axis of the shaft. In this way a stationary image of the scale will be obtained which can be observed by telescope.*

* For the description of this instrument see the Journal "Maschinenbau" 1923–24, p. 1028.

7. Strain Recorders.—In studying the stresses produced in engineering structures or in machine parts during vibrations, the use of special instruments, recording deformations of a very short duration, is necessary. In Fig. 222 below an instrument of this kind, the "Stress Recorder," built by the Cambridge Instrument Company, is shown.* The instrument is especially useful for the measurement of rapidly changing stresses in girders of bridges and other structures under moving or pulsating loads. To find the stress changes in a girder, the instrument

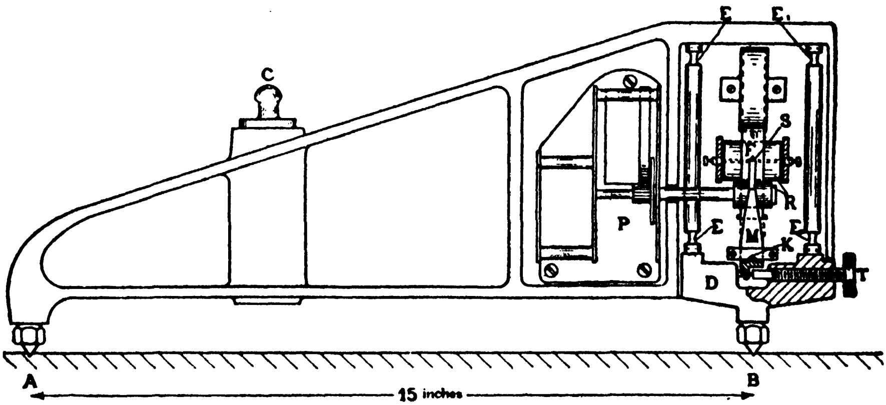

Fig. 222.

is clamped to the girder under test. A clamp is placed upon the projecting part C of a spring plunger, which yields to the clamp so that the instrument is held on to the member, the extension of which is to be measured, by a pre-determined pressure. At one end of the instrument are two fixed points A, while at the other end is a single point carried on the part D, which is free to move in a direction parallel to the length of the instrument. This movement can take place because the bars E and E_1 are reduced at the points marked, the reduction in the size of the bars allowing them to bend at these points, thus forming hinges. The part D is connected to a pivoted lever M carrying the recording stylus S at its upper extremity. Any displacements of the point B due to stress changes in the structure under test are reproduced on a magnified scale by the stylus, and recorded upon a strip of transparent celluloid, which is moved past the stylus by means of a clockwork mechanism P at a rate from about 3 to 20 mm. per second. The mechanical magnification of the

* For the description of this instrument, see "Engineering" (1924), Vol. 118, p. 287.

record in the instrument is ten. The records can be examined by means of a suitable hand microscope, similar to that mentioned on page 449, or direct enlargements from the actual diagrams can be obtained by photographic methods. The record on the film can be read in this manner with an accuracy of .01 mm. Taking the distance between the points A and B equal to 15 inches, we find that the unit elongation can be measured to an accuracy of;

$$\frac{.01}{10 \times 15 \times 25} = 2.66 \times 10^{-6}.$$

For steel this corresponds to a stress of 80 lbs. per square inch. The recording part of the instrument is very rigid and is suitable for vibrations of a very high frequency. For instance, vibrations of a frequency of 1400 per second in a girder have been clearly recorded but this is not necessarily the limit of the instrument. It can be easily attached to almost any part of a structure. The clockwork mechanism driving the celluloid strip is started and stopped either by hand on the instrument itself or by an electrical device controlled automatically or by hand from a distance. The time-marking and position-recording mechanisms are also electrically controlled from a distance. Synchronous readings can be obtained on a number of recorders, as they can be operated from the same time and position signals.

Fig. 223 below represents the diagram of connections of a Magnetic Strain Gauge developed by Westinghouse engineers.* The instrument is held on to the member or girder, the extension of which is to be measured, by clamps such that the two laminated iron U-cores A and B forming a rigid unit are attached to the member at the cross section mm and the laminated iron yoke C through a bar D is attached at the cross section pq so that the gauge length is equal to l. Any changes in the length l due to a change in stress of the member produces relative displacements of C with respect to A and B causing a change in the air gaps. Coils are wound around the two U-shaped iron cores. Through these coils in series an a.c. current is sent of a frequency large with respect to the frequency of the stress variations to be measured. Applying a constant voltage on the two coils in series, the current taken is constant, not dependent on changes in air gap. Unequal air gaps only divide the total voltage in two unequal parts on the two coils. A record of the voltage across one coil is taken by a standard oscillograph. The ordinates of

* Ritter's Extensometer.

the envelope of the diagrams such as that shown in Fig. 219 are proportional to the strains in the member. This magnetic strain gauge was used * for measuring the stresses in rails, produced by a moving locomotive, and proved to be very useful. For a gauge length $l = 8$ in. an accuracy in reading corresponding to a stress of 1000 lbs. per sq. in. can be obtained.

Electric Telemeter.†—This instrument depends upon the well known fact that if a stack of carbon discs is held under pressure a change of

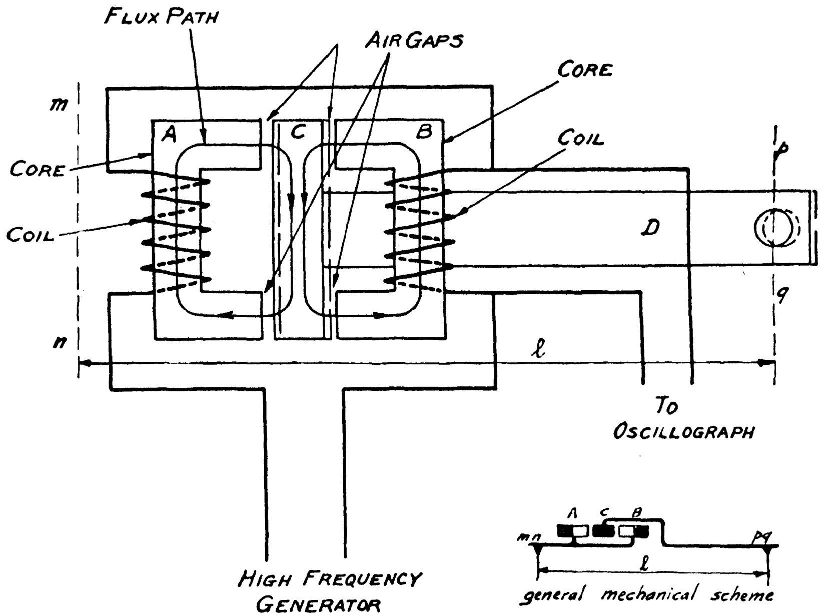

FIG. 223.

pressure will be accompanied by a change in electrical resistance and also a change of length of the stack. The simplest form of the instrument is shown in Fig. 224 when clamped to the member E, the strain in which is to be measured. Any change in distance between the points of support

* See writer's paper presented before the International Congress of Applied Mathematics and Mechanics. Zürich, 1926.

† A complete description of this instrument can be found in the technologic paper of the Bureau of Standards, No. 247, Vol. 17 (1924), p. 737, by O. S. Peters and B. McCollum. See also the paper by O. S. Peters, presented before the Annual Meeting of the American Society for Testing Materials (1927).

A and B produces a change in the initial compression of the stack C of carbon discs, hence a change in the electrical resistance which can be recorded by an oscillograph. Fig. 225 shows in principle the electrical scheme. The instrument 1 is placed in one arm of a Wheatstone bridge, the other three arms of which are 2, 3, and 4. The bridging instrument 5,

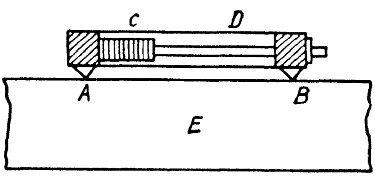

FIG. 224.

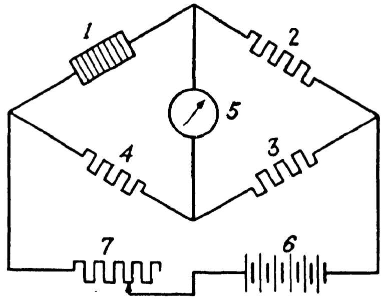

FIG. 225.

which may be a milliammeter or an oscillograph, indicates any unbalance in the bridge circuit. The resistances 2 and 3 are fixed, and 4 is so adjusted that the bridge is balanced when the carbon pile of the instrument is under its initial compression. Any change of this compression, due to strain in test member, will produce unbalance of the bridge, the extent of which will be indicated by the instrument 5, which may be calibrated to read directly the strain in the test member.

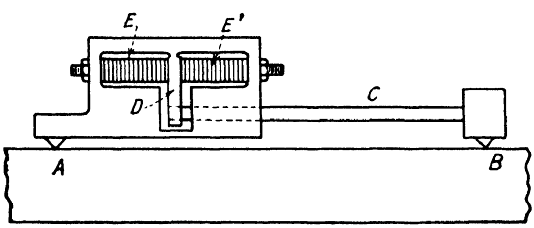

FIG. 226.

An instrument of such a simple form as described above has a defect which grows out of the fact that the resistance of the carbon pile is not a linear function of the displacement. In order to remove this defect, two carbon piles are used in actual instruments (Fig. 226). In this arrangement any change in distance between the points A and B due to strain in the test member will be transmitted by the bar C to the arm D. As a result of this an increase of compression in one of the two carbon piles E and E' and a decrease in the other will be produced. Placing these two carbon piles in the wheatstone bridge as shown in Fig. 227, the effects of changes in the resistance of the two piles will be added and the resultant

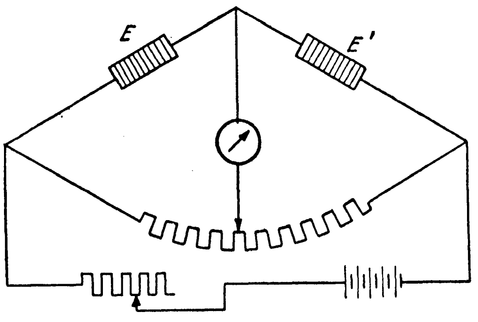

FIG. 227.

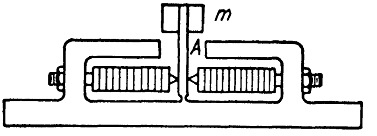

FIG. 228.

effect, which now becomes very nearly proportional to the strain, will be recorded by the bridging instrument.

A great range of sensitivity is possible by varying the total bridge current. Taking this current .6 amp. which is allowed for continuous operation, we obtain full deflection of the bridging instrument with .002

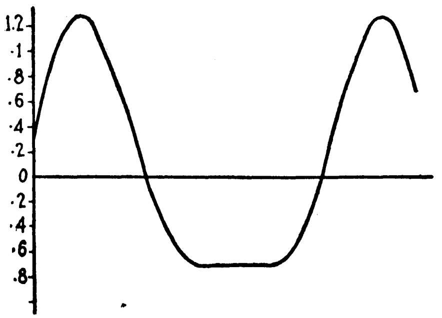

FIG. 229 *a*.

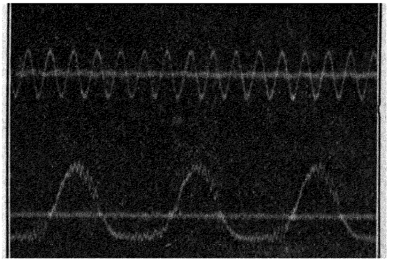

FIG. 229 *b*.

inch displacement. Hence, assuming a gauge length AB (Fig. 226) equal to 8 inches, the full deflection of the instrument for a steel member will represent a stress of about 7500 lbs. per square inch. The instrument proved to be useful in recording the rapidly varying strain in a vibrating member. Vibrations up to more than 800 cycles per second can be reproduced in true proportion.

This instrument has been also successfully used for measuring accelerations.

A slight modification is necessary, consisting of attaching a small mass m to the arm A (Fig. 228). The stacks act as springs, such that the natural period of vibration of the mass m is quite high (of the order of 250 per second in the experiments described below). This instrument was mounted on an oscillating table sliding in guides and operated by a crank. The oscillation of such a table, due to the finite length of the connecting rod is not sinusoidal, but contains also higher harmonics of which the most important is the second. Fig. 229 a shows the acceleration diagram of this table as calculated, and Fig. 229 b gives the oscillograph record obtained from the carbon pile accelerometer mounted on it. The small saw teeth on this diagram have the period of natural vibration of the mass m.

AUTHOR INDEX

SUBJECT INDEX